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Clique Regular Graphs

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ABSTRACT

A maximal complete subgraph of *G* is a clique. The *minimum* (maximum) *clique number* $\vartheta = \vartheta(G)$ $(\omega = \omega(G))$ is the order of a minimum (maximum) clique of *G.* A graph *G* is clique regular if every clique is of the same order. Two vertices are said to dominate each other if they are adjacent. A set S is a dominating set if every vertex in *V*- *S* is dominated by a vertex in *S*. Two vertices are independent if they are not adjacent. The independent domination number $i = i(G)$ is the order of a minimum independent dominating set of *G*. The order of a maximum independent set is the independence number $\beta_0 = \beta_0(G)$. A graph *G* is well covered if $i(G) = \beta_0(G)$. In this paper it is proved that a graph *G* is well covered if and only if $\bar{G}$ is clique regular. We also show that $\vartheta(\bar{G}) = i(G)$.

Keywords: Clique, Minimum clique number, Maximum clique number, Domination number, Well covered graphs and clique regular graphs

INTRODUCTION

All the graphs considered in this paper are finite, simple and undirected. For any undefined terminologies and notations refer to Harary (1969). If a graph *G* is isomorphic to *r* copies of a graph *H,* then we write it as *G* = *r H.* Two vertices are said to dominate each other if they are adjacent. A set $S \subseteq V$ is a dominating set if every vertex in *V*- *S* is dominated by a vertex in *S*. The *domination number* $\gamma = \gamma(G)$ is the order of a minimum dominating set of *G*. The *upper domination number* $\Gamma = \Gamma(G)$ is the maximum order of a minimal dominating set. These concepts of domination are well studied in (Cockayane & Hedetniemi, 1977; Walikar et al., 1979; Haynes et al., 1998; Kamath &Bhat, 2006; Kamath &Bhat, 2007; Bhat et al., 2011; Bhat, Surekha and Bhat, 2011; Bhat et al., 2013; Bhat et al., 2014). The vertex covering

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number $\alpha_0 = \alpha_0(G)$ is the minimum number of vertices needed to cover all the edges of a graph while independence number $\beta_0 = \beta_0(G)$ is the maximum number of vertices in an independent set of *G*. These two numbers are related by classical Gallai's Theorem: $\alpha_0(G) + \beta_0(G) = p$. The *upper vertex covering number* $\epsilon = \epsilon(G)$ is the maximum order of a minimal covering of *G*. The *independent domination number* $i = i(G)$ is the minimum order of an independent dominating set of G. Naturally, we have an extension of Gallai's theorem to these numbers as: $\epsilon(G) + i(G) = p$. A maximal complete subgraph is a *clique.* The minimum number of cliques (not necessarily maximal) that cover all the vertices of a graph is well known in graph theory as *partition number* $\theta_0 = \theta_0(G)$ introduced by (Berge, 1962) and has been celebrated in Berge's conjecture on perfect graphs. Choudam et al. (1975) studied its edge analogue line *clique covering number* $\theta_1(G)$ defined as the minimum number of cliques that cover all the lines of a graph. The minimum number of colours needed to properly colour the vertices of *G* is the *chromatic number* $\chi = \chi(G)$. Since independent sets and cliques exchange their properties on complementation $\theta_0(G) = \chi(\bar{G})$. Bhat et al. (2013) defined block domination parameters and studied their relationship between other domination parameters. In this paper we obtain few bounds on minimum clique number and characterized well covered graphs using clique regular graphs.

MINIMUM CLIQUE NUMBER

The *minimum clique number* $\vartheta(G)$ is the order of a minimum clique of *G while* the *maximum clique number* $\omega(G)$ is the order of a maximum clique of *G*. It is immediate that $\vartheta(G) \leq \omega(G)$. Even though these two parameters are well studied in literature, the first parameter *minimum clique number* $\vartheta(G)$ received less attention and we are interested in it than the later. If *G* has an isolated vertex, then $\vartheta(G) = 1$. If *G* is a triangle free graph without isolates, then $\vartheta(G) = 2$. The girth $g(G)$ of a graph is the length of the shortest cycle in *G*. Girth of a graph is defined if *G* has a cycle otherwise we define $g(G) = \infty$. Since girth of any graph is at least 3, $\vartheta(G) \leq g(G)$ if $\vartheta(G) \leq 3$. Moreover, $\vartheta(G) \geq 4$ if then every minimum clique contains a triangle and hence $g(G) = 3 < 4 \leq \vartheta(G)$. It is well known that $\omega(G) = \beta_0(\bar{G})$. A similar result for minimum clique number is obtained in the next proposition.

Proposition 1 *For any graph G,*

$$\vartheta(G) = i(\bar{G})$$

Proof. Let $\vartheta(G) = k$, and S be the set of vertices of a minimum clique of G. Since independent sets and cliques exchange their properties on complementation, S forms a maximal independent set with minimum number of vertices in $\bar{G}$. Then by Ore's theorem (Ore, 1962), we have every maximal independent set is a minimal dominating set. Therefore, S is a minimum independent dominating set of $\bar{G}$. Hence $\vartheta(G) = k = |S| = i(\bar{G})$.

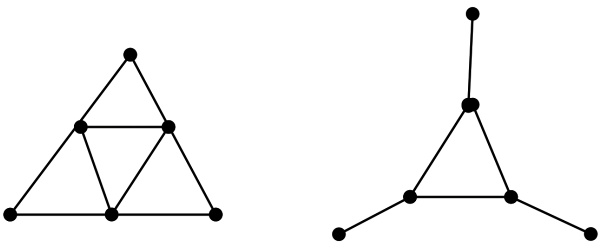

Figure 1. Hajo's Graph G and its compliment (Haynes et al.,1998)

We can observe that for the Hajo's graph *G* in Figure 1, $\vartheta(G) = 3$ and for the compliment of Hajo's graph $i(\bar{G}) = 3$.

The doma*tic number* (*G*) is the maximum order of a partition of vertex set in to dominating sets. As $\omega(G)$ served a best lower bound for chromatic number is evident from the known inequality, $\omega(G) \leq \chi(G) \leq 1 + \Delta(G)$ (Cockayane &Hedetniemi, 1977, p. 250). A clique graph $K_G(G)$ of *G* is a graph with vertex set as cliques of *G* and any two vertices in $K_G(G)$ are adjacent if and only if the corresponding cliques in *G* have a vertex in common. Independence graph *I*(*G*) is a graph with vertex set as set of all maximal independent sets of *G* and any two vertices in *I*(*G*) are adjacent if they have a vertex in common. We observe that any maximal independent set in *G* is a clique in $\bar{G}$ and vice versa. Hence $K_G(\bar{G}) \cong I(G)$. Cockayne &Hedetniemi (1977, p. 257) proved that if $K_G(G)$ is an even cycle, then $\vartheta(G) \leq d(G)$. Zelinca (1981), constructively shown that the analogous assertion is false if $K_G(G)$ is an odd cycle. Hence $\vartheta(G)$ can exceed the domatic number. Thus $\vartheta(G)$ and $d(G)$ are incomparable. We now provide an upper bound for minimum clique number in terms of minimum degree and order of *G*. We use the following notations. Let $N(v) = \{u\epsilon V | u \text{ is adjacent tov}\}$ and $N[v] = N(v) \cup \{v\}$. Then $\langle N[v] \rangle$ denote the subgraph induced by the set N[v]. Let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of G while $\bar{\delta} = \delta(\bar{G})$and $\bar{\Delta} = \Delta(\bar{G})$. Let $V_\delta = \{v\epsilon V | \deg(v) = \delta\}$.

Proposition 2 *For any graph G with minimum degree* $\delta(G)$*,*

$$\vartheta(G) \leq 1 + \delta(G).$$

Further, the equality holds if and only if $\langle N[v] \rangle$ *is a minimum clique of G for every* $v\epsilon V_\delta$

Proof. We first note that $\Delta + \bar{\delta} = \bar{\Delta} + \delta = p - 1$. It is well known that $i(G) \leq p - \Delta(G)$ (Haynes et al., 1998, p. 312). Therefore $\vartheta(G) = i(\bar{G}) \leq p - \bar{\Delta} = 1 + \delta(G)$.

Suppose that $\vartheta(G) = 1 + \delta(G)$. Then if $\langle N[v] \rangle$ is not a minimum clique of *G* for some $v\epsilon V_\delta$ then $\vartheta(G) < |\langle N[v] \rangle| = 1 + \delta(G)$ a contradiction.

Converse is straight forward and we omit the proof.

The bound is sharp for the complete graph K_n and star graph $K_{1,n}$.

The following results relate the different graph parameters which appears in (Haynes et al., 1998, p.374).

Proposition 3. *For any graph G*

$$\frac{p}{1+\Delta} \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \qquad [1]$$

On complementing the result [1], we get the next corollary and one can see that the $\vartheta(G)$ fits best in between the known graph parameters.

Corollary 3.1 *For any graph G,*

$$\frac{p}{p-\delta} \leq \gamma(\bar{G}) \leq \vartheta(G) \leq \omega(G) \leq \Gamma(\bar{G}). \qquad [2]$$

The idiomatic number $d_i = d_i(G)$ is the maximum order of partition of vertex set in to independent dominating sets. The idiomatic number does not exist for all graphs. A graph *G* is indominable if *G* admits an independent dominating set partition. The maximal clique partition number $\theta_m = \theta_m(G)$ is the maximum order of partition of vertex set in to cliques of *G*. A graph which admits a clique partition is called clique partitionable. Hence $\theta_m(G) = d_i(\bar{G})$. If *G* is indominable then $\bar{G}$ is clique partitionable. If both *G* and $\bar{G}$ are indominable then *G* is called strongly indominable. We now provide an upper bound to domination number of an indominable graph in terms of minimum clique number.

Proposition 4. *If G or* $\bar{G}$ *is indominable*

Proof. If *G* or $\bar{G}$ is indominable, it is proved that $\vartheta(G) \leq d_i(G) \leq d(G)$ (Walikar et al., 1979, p.109). Therefore $\gamma(G)\vartheta(G) \leq i(G)\vartheta(G) \leq i(G)\, d_i(G) \leq p$. This yields the desired inequality.

The bound is attained for any even cycle, regular bipartite graph or complete graph.

Corollary 4.1 *If G is clique partitionable then* $i(G) \leq d_i(\bar{G}) = \theta_m(G)$- *is the partition vertex set in to maximal cliques of G.*

Walikar et al. (1979) has proved that for any cubic graph, if there exists a maximal clique of order 2 then $\gamma(\bar{G}) = 2$. We now prove a stronger result with much more ease and the above result is a corollary to the next proposition.

Proposition 5. *For any graph G with,* $\vartheta(G) = 2$ *then* $\gamma(\bar{G}) = i(\bar{G}) = 2$

Proof. From Proposition 1, we have $2 = \vartheta(G) = i(\bar{G})$. As every independent dominating set is a dominating set we have $\gamma(\bar{G}) \leq i(\bar{G})$. Suppose $\gamma(\bar{G}) < i(\bar{G})$then $\gamma(\bar{G}) = 1$. As any singleton set is independent we then have $i(\bar{G})$. This is a contradiction to the statement that $i(\bar{G}) = 2$. Therefore $\gamma(\bar{G}) = i(\bar{G})$.

Corollary 5.1. *If G is a cubic graph with a maximal clique of order 2, then* $\gamma(\bar{G}) = 2$.

Proof. Since *G* is cubic and there exists a maximal clique of order 2 together implies that . Then the result follows by Proposition 2.5.

CLIQUE REGULAR GRAPHS

The concept of well covered graphs is studied in (Plummer, 1970; Plummer, 1993; Dean &

Zeto, 1994; Ravindra, 1997). A graph *G* is *well covered* if every maximal independent set is of same order. In other words, *G* is well covered if and only if $i(G) = \beta_0(G)$. Equivalently, $\epsilon(G) = \alpha_0(G)$.

The above definition motivated the description of another special class of graphs called clique regular graphs. A graph *G* is *clique regular* if every clique is of same order. Thus *G* is k- *clique regular* graph if $\omega(G) = \vartheta(G) = k$. For example, 3-clique regular graph and 5-clique regular graphs are shown in the Figure 2.

Remark 1. The maximum number of vertices in a minimal vertex cover is called maximum vertex covering number $\varepsilon(G)$. It is proved that for any graph *G,* $\varepsilon(G) + i(G) = p$ (Haynes, et al.,1998, p. 524). Using Proposition 5, this result can now be restated as $\varepsilon(G) + \vartheta(\bar{G}) = p$ or equivalently, $\varepsilon(\bar{G}) + \vartheta(G) = p$.

Remark 2. Similarly, from the above Remark 1, we may write $\alpha_0(G) + \beta_0(G) = p$ (Gallai's Theorem) as $\alpha_0(G) + \omega(\bar{G}) = p$ or equivalently, $\alpha_0(\bar{G}) + \omega(G) = p$.

Example 1.

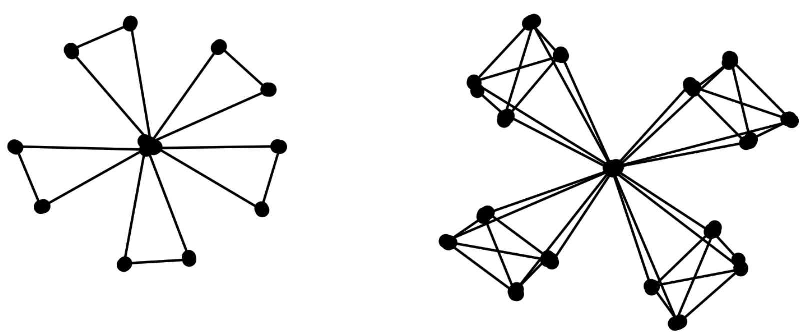

Figure 2. A 3-clique regular and 5-clique regular graphs (Haynes et al.,1998)

The advantage of knowing $\vartheta(G)$ and $\omega(G)$ is that one can easily determine the independent domination number and independence number of $\bar{G}$. Using this technique $i(\bar{G})$ and $\beta_0(\bar{G})$ for some standard graphs are obtained for some standard graph. A double star is a tree $T = K_{1,n} * K_{1,m}$ obtained by joining the two nonpendant vertices of $K_{1,n}$ and $K_{1,m}$.

Proposition 6

(i) *For any double star* $T = K_{1,n} * K_{1,m}$,
$i(\bar{T}) = 2 = \beta_0(\bar{T})$; $\vartheta(\bar{T}) = \min(m+1, n+1)$ and $\omega(\bar{T}) = m + n$

(ii) *For any tree* *T*, $i(\bar{T}) = 2 = \beta_0(\bar{T})$.

(iii) *For any triangle free graph*, $i(\bar{G}) = 2 = \beta_0(\bar{G})$.

Proof. Note that for any double star, $T = K_{1,n} * K_{1,m}$, $i(T) = \min(m+1, n+1)$, $\beta_0(T) = m + n$ *and* $\vartheta(T) = \omega(T) = 2$. Then the result (i) follows from Proposition 5 and Remark 1. The rest of the results can be proved similarly.

As any Cube graph Q_n, Petersens graph, Hexagonal hub graph in Figure 3 are triangle free graphs without isolates and hence the next corollary.

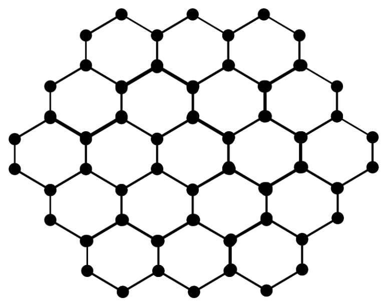

Figure 3. Hexagonal hub graph H (Haynes et al.,1998)

Corollary 6.1

(i) If G is a Petersen's graph, then $i(\bar{G}) = 2 = \beta_0(\bar{G})$

(ii) *For any cube graph* Q_n, $i(\bar{Q}_n) = 2 = \beta_0(\bar{Q}_n)$

(iii) *For the hexagonal hub graph H,* $i(\bar{H}) = 2 = \beta_0(\bar{H})$

(iv) *For any grid graph G,* $i(\bar{G}) = 2 = \beta_0(\bar{G})$

(v) *For any wheel graph*, W_n, $i(\bar{W}_n) = \beta_0(\bar{W}_n) = 4, if\ n = 4$ *and* $i(\bar{W}_n) = \beta_0(\bar{W}_n) = 3,\ if\ n \geq 5$

Proposition 7 *For any graph G*,

(i) $\beta_0\left(\overline{B_G(G)}\right) = \Delta_{vb}(G) = \Delta_{bv}\left(B_G(G)\right)$

(ii) $i\left(\overline{B_G(G)}\right) = \delta_{cvb}(G) = \delta_{bv}\left(B_G(G)\right)$

(iii) $i\left(\overline{C_G(G)}\right) = \delta_{NPc}(G) = \delta_{cvb}(B_G(G))$

(iv) $\beta_0(\overline{C_G(G)}) = \Delta_c(G) = \Delta_{cvb}(B_G(G))$

Proof. For any graph *G*, every block of the block graph $B_G(G)$ is a clique. Since all the blocks incident on a cut vertex of *G* are mutually adjacent, these blocks form a clique in $B_G(G)$. Therefore, number of blocks incident on a cut vertex $= d_{vb}(G) =$ number of vertices in the corresponding block in $B_G(G) = d_{bv}\left(B_G(G)\right)$. Hence $\Delta_{vb}(G) = \Delta_{bv}\left(B_G(G)\right) = \omega\left(B_G(G)\right)$. Similarly, $\delta_{cvb}(G) = \delta_{bv}\left(B_G(G)\right) = \vartheta\left(B_G(G)\right)$. Then the results (i) and (ii) follow by Proposition 5 and Remark 1.

Again, for any graph *G*, every block of the cutvertex graph $C_G(G)$ is a clique. Since all cutvertices in a nonpendant block are mutually adjacent, these cutvertices form a clique in $C_G(G)$. Therefore, number of cutvertices incident on a block $= d_c(G) =$ number of vertices in the corresponding block in $C_G(G) = d_{bv}(C_G(G))$. Hence $\Delta_c(G) = \Delta_{bv}(C_G(G)) = \omega(C_G(G))$. Similarly, $\delta_{NPc}(G) = \delta_{bv}(C_G(G)) = \vartheta(C_G(G))$. Then the results (iii) and (iv) follow by Proposition 5 and Remark 1.

Proposition 8 *For any graph G*, with maximum degree $\Delta(G)$ *and minimum degree* $\delta(G)$,

(i) $\beta_0(\overline{L(G)}) = \Delta(G)$

(ii) $i(\overline{L(G)}) = \delta(G)$

Proof. Let *v* be a vertex of maximum degree $\Delta(G)$ and x be an edge containing the vertex *v*. Then all the $\Delta(G)$ edges incident on v are mutually adjacent and hence form a maximum clique of order $\Delta(G)$ in *L*(*G*). Hence $\omega(L(G)) = \Delta(G)$. The result (ii) follows by Proposition 5 and Remark 1.

The following corollaries are immediate from the above proposition.

Corollary 8.1 *If G is regular then L*(*G*) *is clique regular*

Corollary 8.2 *If G is regular then complement of L*(*G*) *is well covered*

DISCUSSION AND CONCLUSION

A graph is regular if every vertex is of same degree. This class of graphs are well studied in literature. Here we have introduced and studied a new class of graphs called clique regular graphs. It is observed that every regular graph need not be clique regular and every clique regular graph need not be regular. The properties of clique regular graphs can be studied in depth as future work. The effect of regular cliques in *G* can be extended to Line graph, Block graph and clique graphs.

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