BAYESIAN APPROACH FOR ESTIMATING THE PARAMETERS AND PERCENTILES OF THE TIME-TO-FAILURE DISTRIBUTION BASED ON GENERAL DEGRADATION MODELS

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PENDEKATAN BAYESAN UNTUK MENGANGGARKAN PARAMETER DAN PERSENTIL TABURAN MASA KEGAGALAN BERDASARKAN MODEL DEGRADASI UMUM

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DECLARATION

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ABSTRAK

Model degradasi sering digunakan pada data degradasi untuk mengkaji taburan masa kegagalan. Dalam kajian ini, pendekatan Bayesan digunakan pada tiga jenis model degradasi yang berbeza, iaitu model degradasi linear, eksponen dan kuasa, untuk menganggarkan parameter taburan masa kegagalan dan persentilnya. Dua taburan yang berbeza diandaikan untuk parameter degradasi model tersebut. Pertama, parameter degradasi diandaikan mengikut taburan normal pencong dengan tiga parameter taburan bercantum tidak bersandar, yang mana prior gamma diandaikan untuk parameter bentuk, sementara parameter skala dan lokasi diandaikan bertabur mengikut taburan seragam. Taburan kedua yang diandaikan untuk parameter degradasi adalah taburan log-logistik dengan dua parameter rawak bercantum yang tidak bersandar, yang mana parameter bentuk diandaikan mengikut taburan gamma, sementara parameter skala diandaikan mengikut taburan seragam. Berdasarkan kaedah pensampelan Gibbs yang dijalankan menggunakan platform JAGS, model yang dipertimbangkan diaplikasikan kepada data simulasi dan data sebenar, dan hasil yang didapati telah dibandingkan berdasarkan anggaran titik, kepincangan, sisihan piawai dan kriteria maklumat devians. Dalam pemodelan degradasi Bayesan berdasarkan semua model yang dikaji, didapati bahawa pemodelan yang melibatkan parameter degradasi normal pencong adalah lebih baik berbanding pemodelan yang melibatkan parameter log-logistik berdasarkan nilai sisihan piawai dan kriteria maklumat devians yang lebih kecil.

ABSTRACT

The degradation models are often applied on the degradation data for studying time-to-failure distribution. In this study, the Bayesian approach is applied on the three different types of degradation models, which are linear, exponential and power degradation models, for estimating the parameters of the time-to-failure distribution and its percentiles. Two different distributions are assumed for the degradation parameter of the models. The degradation parameter is firstly assumed to follow the skew-normal distribution with three jointly independently distributed parameters such that the gamma prior is assumed for the shape parameter, while the scale and the location parameters are assumed uniform. The second distribution assumed for the degradation parameter is the log-logistic distribution with two jointly independent random parameters where the shape parameter is assumed gamma, while the scale parameter is assumed uniform. Based on the Gibbs sampling method carried out under the JAGS platform, the models considered are applied on the simulated data and the real data, and the results found are compared in terms of point estimate, biasness, standard deviation and deviance information criteria. In the Bayesian degradation modelling based on all the models studied, it is found that modelling involving the skew-normal degradation parameter outperformed modelling involving the log-logistic parameter based on smaller values of standard deviation and deviance information criteria.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

TTF	Time-to-Failure
MCMC	Markov Chain Monte-Carlo
JAGS	Just Another Gibbs Sampling
NTJE	NASA Turbofan Jet Engine
HI	Health index
RUL	Remaining useful life
SN	Skew-normal distribution
T	Owen's T function
TTFL	Time-to-Failure distribution for Linear Degradation Model
cdf	Cumulative distribution function
pdf	Probability density function
PE	Estimated parameters and Percentiles
B	Bias
SD	Standard deviation
$\hat{R}$	convergence statistics
psrf	potential scale reduction factor
DIC	Deviance information criterion
TTFP	Time-to-Failure distribution for Power Degradation Model
SE	Standard error
TTFE	Time-to-Failure distribution for Exponential Degradation Model

LIST OF SYMBOLS

y_{ij}	Observed degradation measurement of i th unit at time t_{ij}
$D(t_{ij};\varphi,\beta_i)$	Actual path of i th unit at time t_{ij},
φ	Fixed-effect parameter
β_i	Degradation parameter for i th unit
ε_{ij}	Random error term
σ_ε^2	Variance of random error term
n	Sample size
m_i	total number of observations on $i\ th$ unit.
D_f	critical degradation level
t	time-to-failure
φ_o	Fixed effect parameter
$g_X(.)$	Probability density function of random variable X
$\phi(.)$	probability density function of standard normal distribution
$\Phi(.)$	cumulative distribution function of standard normal distribution
λ	Shape or skewness parameter of skew-normal distribution
σ^2	Variance
μ	location parameter of skew-normal distribution
σ	scale parameter of skew-normal distribution
$G_\beta(.)$	Cumulative distribution function of β
$g_\beta(.)$	Probability density function of β
α	scale parameter of log-logistic distribution
ω	shape parameter of log-logistic distribution
$\boldsymbol{\theta}$	Vector of parameters
$q(\boldsymbol{\theta})$	Probability distribution of $\boldsymbol{\theta}$
$f(t,\boldsymbol{\theta})$	The joint probability distribution of t and $\boldsymbol{\theta}$
$m(t)$	The marginal distribution of t

$\pi(\boldsymbol{\theta}\|\boldsymbol{t})$	Posterior distribution of $\boldsymbol{\theta}$
$L(\boldsymbol{\theta};\boldsymbol{t})$	The likelihood function of $\boldsymbol{\theta}$
$q_i(\theta_i)$	The prior distribution of θ_i
$F_{T-L}(.)$	Cumulative distribution function of time-to-failure distribution for linear degradation model
$f_{T-L}(.)$	Probability density function of time-to-failure distribution for linear degradation model
$F_{T-LSN}(.)$	Cumulative distribution function of time-to-failure distribution under linear degradation model with skew-normal degradation parameter
$f_{T-LSN}(t)$	Probability density function of time-to-failure distribution under linear degradation model with skew-normal degradation parameter
r	The probability values of the percentiles
t_{r-LSN}	The 100 r^{th} percentile of time-to-failure distribution under linear degradation model with skew-normal degradation parameter
$t_1, t_2, \ldots, t_n$	Random sample of size *n* from the time to failure distribution
$F_{T-LL}(.)$	Cumulative distribution function of time-to-failure distribution under linear degradation model with log-logistic degradation parameter
$f_{T-LL}(.)$	Probability density function of time-to-failure distribution under linear degradation model with log-logistic degradation parameter
t_{r-LL}	The 100 r^{th} percentile of time-to-failure distribution under linear degradation model with log-logistic degradation parameter
$\pi_{LSN}(\lambda,\sigma,\mu,\varphi\|\boldsymbol{t})$	The posterior distribution of the TTFL with skew-normal degradation parameter
$\pi_{LL}(\alpha,\omega,\varphi\|\boldsymbol{t})$	The posterior distribution of the TTFL with log-logistic degradation parameter
$f(\lambda\|\boldsymbol{t})$	The marginal posterior density of λ given $\boldsymbol{t}$
$\overline{D}(\boldsymbol{\theta})$	Mean posterior deviance
P_D	The effective number of parameters of the model
$F_{T-P}(.)$	Cumulative distribution function of time-to-failure distribution under power degradation model

$f_{T-P}(.)$	Probability density function of time-to-failure distribution under power degradation model
$F_{T-PSN}(.)$	Cumulative distribution function of time-to-failure distribution under power degradation model with skew-normal degradation parameter
$f_{T-PSN}(.)$	Probability density function of time-to-failure distribution under power degradation model with skew-normal degradation parameter
t_{r-PSN}	100 $r\ th$ percentile of the time-to-failure distribution under power degradation model with skew normal degradation parameter.
$F_{T-PL}(.)$	Cumulative distribution function of time-to-failure distribution based on the power degradation model with log-logistic degradation parameter
$f_{T-PL}(.)$	Probability density function of time-to-failure distribution under the power degradation model with log-logistic degradation parameter
t_{r-PL}	100 r^{th} percentile of the time-to-failure distribution under power degradation model with log-logistic degradation parameter.
π_{PSN}	The posterior distribution of the TTFP with skew-normal degradation parameter
$\pi(\mu\|\boldsymbol{t})$	The marginal posterior density function of μ
$\pi(\sigma\|\boldsymbol{t})$	The marginal posterior density function of σ
$\pi(\lambda\|\boldsymbol{t})$	The marginal posterior density function of λ
$\pi(\varphi\|\boldsymbol{t})$	The marginal posterior density function of φ
π_{PL}	The posterior distribution of the TTFP with log-logistic degradation parameter
$F_{T-E}(.)$	Distribution function of the time-to-failure distribution based on the exponential degradation model
$f_{T-E}(.)$	Probability density function of time-to-failure distribution under the exponential degradation model with log-logistic degradation parameter
$F_{T-ESN}(.)$	Distribution function of time-to-failure distribution based on the exponential degradation model with skew normal degradation parameter

$f_{T-ESN}(.)$	Probability density function of time-to-failure distribution based on the exponential degradation model with skew normal degradation parameter
t_{r-ESN}	100 $r\ th$ percentile of the time-to-failure distribution under exponential degradation model with skew normal degradation parameter.
$F_{T-EL}(.)$	Probability distribution function of time-to-failure distribution under the exponential degradation model with log-logistic degradation parameter
$f_{T-EL}(.)$	Probability density function of time-to-failure distribution under the exponential degradation model with log-logistic degradation parameter
t_{r-EL}	100 r^{th} percentile of the time-to-failure distribution under exponential degradation model with log-logistic degradation parameter.
π_{ESN}	The posterior distribution of the TTFE with skew-normal degradation parameter
π_{EL}	The posterior distribution of the TTFE with log-logistic degradation parameter

CHAPTER I

INTRODUCTION

1.1 INTRODUCTION

When conducting any daily work, people usually rely on reliable equipment or tools which function properly and continue performing consistently well for a substantial period of time during usage. A reliable product can only come by when both the manufacturers and the consumers are concerned with certain characteristics of the product such as quality. Accordingly, reliability which is the essence of quality, is always being an important subject matter in many areas of research. Meeker and Escobar (1998), for example, have defined reliability of a unit as the probability that a unit will perform its intended function until a specified point of time under encountered use conditions.

Reliability, a fundamental concept in engineering, concerns with the ability of a system or component to perform its required function under stated conditions for a specified period of time (Jiang 2015). Traditional work in reliability involves the application of lifetime data for assessing reliability. Lifetime data refers to the duration of time until a specific event of interest such as failure occurs (Schober & Vetter 2018). Accordingly, one measured item contributes to only one observation of failure time data. Collecting data in this way is time consuming, hard and expensive when one deals with expensive items such as jet engine turbofan. In the last two decades, degradation data is commonly used to assess the reliability, as reported in the works such as the works by Chen et al. (2024), Gorjian et al. (2010) and Li et al. (2020), which involves measurements of a performance characteristic that deteriorates over time such as wear, corrosion or capacity loss. In the degradation process, multiple observations can be obtained based on one subject and this data can be used to estimate time-to-failure (TTF) distribution (Gorjian et al. 2010).

In the context of a degradation model, the TTF is often influenced by a gradual deterioration or degradation of the system over time (Freitas et al. 2010). This degradation can be due to various factors, such as wear, fatigue, corrosion or other physical processes. There are many common examples of degradation models, including Wiener process, Gamma process, Inverse Gaussian process and Markov chain models (Chen et al. 2020c; Kharoufeh & Cox 2005; Li et al. 2018). One of the most popular degradation models which is widely used to estimate TTF is the general degradation model.

The general degradation model assumes that the performance of a system or condition can be characterized by a degradation process, which can be described by a random variable or a stochastic process (Lu et al. 2021). Information available based on this degradation process can be used to model the gradual deterioration of the system over time and to estimate the TTF distribution. The TTF distribution under a general degradation model can be derived using the parameters of the underlying degradation process. This distribution can provide valuable information about the reliability and expected lifetime of the system, which is crucial for maintenance planning, design optimization and risk assessment.

The analysis of TTF distribution under a general degradation model often involves the use of advanced mathematical and statistical techniques, such as stochastic calculus, reliability theory, and statistical inference. Researchers and practitioners in the field of reliability engineering have developed various analytical and numerical methods to model and analyse the TTF distribution in the context of general degradation models such as Monte-Carlo simulation method and Bayesian inference.

1.2 RESEARCH PROBLEM

It is pertinent to know how the functioning of a particular product deteriorates over time. This deterioration which is given the term degradation can be studied based on the information available with the degradation data. Determining the statistical model based on degradation data, although is quite challenging, is beneficial for describing the underlying degradation process.

A type of degradation model which is commonly applied in modelling the degradation data is the general degradation path model (Ye & Xie 2015). Mathematical functions which are usually considered under this model are linear or non-linear functions (Rawashdeh et al. 2018b; Siju & Kumar 2018). In the case of non-linear functions, exponential and power functions are commonly applied in estimation of TTF distribution. The general degradation path model is employed by statisticians to estimate TTF distribution and its percentiles based on parametric, semi-parametric or non-parametric methods.

In the field of reliability engineering and survival analysis, estimating the TTF distribution of components, systems or processes accurately is critical to determine the maintenance planning and design optimization (Schober & Vetter 2018). Traditional methods such as maximum likelihood estimation and ordinary least squares estimation often depend on normal or exponential distributions to model failure times (Ba Dakhn et al. 2017). However, real-world degradation data frequently exhibit the properties of skewness and kurtosis that these conventional distributions fail to hold adequately. Many studies have been carried out to address the issue of asymmetric properties of degradation data (Ghaderinezhad et al. 2020b; Mameli et al. 2012). Since there are not many studies which consider the usage of skew-normal in the general degradation modelling, this study embarks on the application of skew-normal distribution due to the flexibility brought about by the distribution.

The skew-normal distribution, introduced by Azzalini (1985), offers a more flexible approach to modelling of asymmetric data. Unlike the normal distribution, which is symmetric around its mean, the skew-normal distribution incorporates a shape parameter that allows it to model skew data. This additional parameter provides a greater adaptability, enabling the skew-normal distribution to fit a wider range of data patterns. In the context of TTF analysis, it is believed that this flexibility can lead to a more precise estimation, especially when dealing with datasets that exhibit pronounced skewness. When the shape parameter is zero, while retaining its properties the skew-normal distribution reduces to the normal distribution. This ensures continuity of the usage of the skew-normal distribution in the analysis without abandoning the familiar framework of the normal distribution.

In practical applications, the skew-normal distribution has shown some promise in various fields. For example, in the aerospace industry, where the knowledge on reliability of components is critical for determining the maintenance planning, skew-normal models have been employed to estimate the TTF of jet engine parts (Cox & Oakes 1984). Similarly, in the medical field, skew-normal distribution has been used to model the survival times of patients undergoing specific treatments; thus, providing a more accurate prognostic information (Collett 2015). These examples underscore the application of skew-normal distribution across diverse domains, emphasizing its potential to enhance reliability assessments and decision-making processes. By that, we can say that the skew-normal distribution represents a significant advancement in the estimation of TTF distributions. Its ability to model asymmetric data with greater accuracy than traditional distributions make it an asset in reliability engineering and survival analysis.

An important method in statistical inference which employs prior and experimental information in order to draw inference about a population interest is the Bayesian method. The Bayesian method is now becoming widely accepted to solve applied statistical problems in the area of econometrics, education, engineering and medicine (Abong et al. 2017; Cauchi et al. 2017; Fernandes et al. 2022; Wu et al. 2019a). This method has been used extensively in literature throughout the last two decades due to the discovery of Markov Chain Monte-Carlo (MCMC) method which is a computational tool to deal with posterior distribution that is mathematically complex. Accordingly, Bayesian approach is becoming versatile in the analysis of degradation data.

In this study, the flexibility of the skew-normal distribution with the presence of location, scale and shape parameters is considered for estimating the TTF distribution based on the general degradation model. In addition, the flexibility of the

Bayesian approach which brings in extra information through the prior distribution has made the approach as the choice for the method of estimation of the parameters of TTF distribution and its percentiles (Fúquene Patiño et al. 2018; Grzenda 2016; Lemoine 2019). The Bayesian approach is used to estimate the parameters and the percentiles of the TTF distribution according to three general degradation models which are linear, power and exponential. According to the promising result in term of parameter estimation in the linear degradation modelling involving the degradation parameter following the log-logistic as reported by Rawashdeh et al. (2018), in this study, the performance of skew-normal general degradation models is compared with log-logistic general degradation models. Additionally, applications of real data are provided.

1.3 STUDY OBJECTIVES

Based on the aforementioned research background, the objectives of this study are as follows:

i) To derive the TTF distribution based on linear, power and exponential degradation models where the degradation parameter is assumed to follow either the skew-normal or the log-logistic distribution.

ii) To estimate the parameters of the TTF distributions derived and their percentiles based on the Bayesian method.

iii) To study the performance of the Bayesian method under different assumptions of priors using simulation study as well as some real data applications.

iv) To compare the results obtained based on the Bayesian approach for linear, power and exponential degradation models with skew-normal degradation parameter with those found involving log-logistic degradation parameter under simulated data and real datasets.

1.4 SCOPE OF STUDY

The area of focus in this study is the application of the general degradation models which are linear, power and exponential for estimating the TTF distribution. Under the parametric method, the degradation parameter of the proposed models is assumed to follow either skew-normal or log-logistic distributions. Then, the Bayesian approach is applied to estimate the parameters of the TTF distributions based on the simulated and real data.

1.5 FLOWCHART OF THE ANALYSIS

In this section, the steps of analysis based on the linear, power and exponential general degradation models are provided according to the objectives of our thesis as follows:

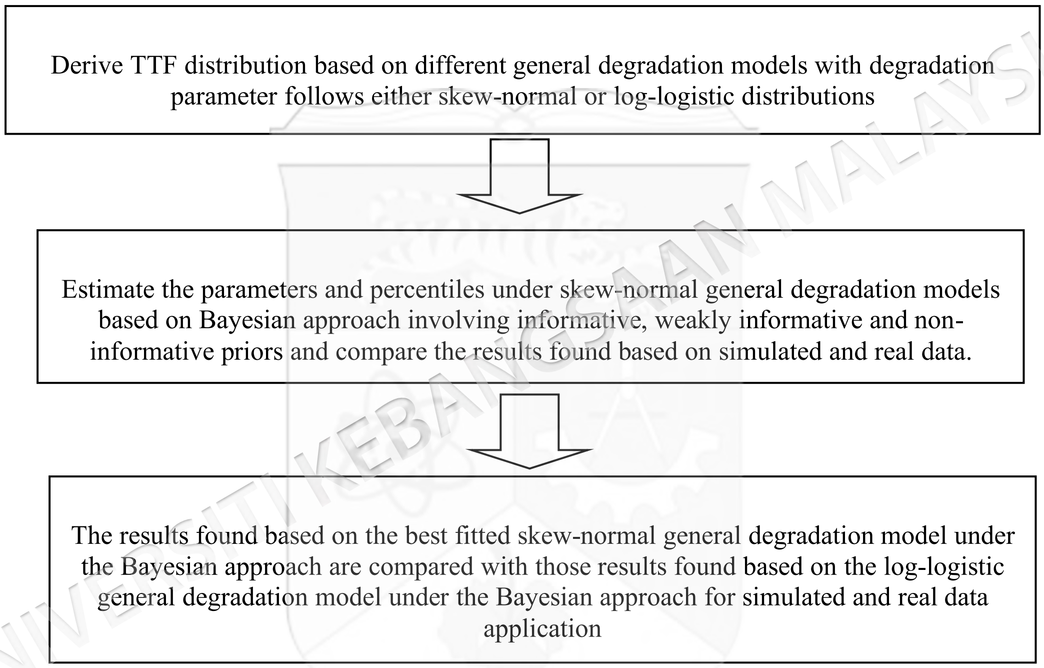

Figure 1.1 Flowchart of the analysis in the study method

1.6 STRUCTURE OF THE THESIS

The material of this study is presented in the following chapters. Chapter 2 discusses general degradation models, TTF distribution, skew-normal distribution, log-logistic distribution and Bayesian method. In addition, some historical background and applications are provided.

In Chapter 3, a linear degradation model is presented. Derivation of the TTF distribution under the presence of skew-normal and log-logistic degradation parameters is given. In addition, the Bayesian approach is applied to estimate the parameters and percentiles of the TTF distribution with the allowance for the skew-normal degradation parameter following several different prior distributions. This estimation is implemented by using MCMC method under JAGS platform. A comparison is made between log-logistic linear degradation modelling and skew-normal linear degradation modelling under the Bayesian approach. The fitting of the Bayesian models for the GaAs laser degradation data is discussed with reference to convergence analysis. The conclusion of this chapter is provided.

In Chapter 4, a power degradation model assuming that the degradation parameter follows either skew-normal distribution or log-logistic distribution is applied. The TTF distribution is derived. Also, the Bayesian approach is used to estimate the parameters and percentiles of the TTF distribution involving several types of priors. A comparison is carried out between the skew-normal power degradation model and the log-logistic power degradation model. A real dataset which consists of NASA turbofan jet engine (NTJE) is considered for the two models. Based on the convergence diagnostic found using the JAGS platform, results on the parameter estimation are discussed. A summary of Chapter 4 is provided.

In Chapter 5, the TTF distribution is derived based on exponential degradation model with the assumption of degradation parameter follows either skew-normal distribution or log-logistic distribution. Based on the simulated data, the parameters and the percentiles of the TTF distribution are estimated by using the Bayesian approach involving various choices of prior distributions. Here, a comparison of the Bayesian results between the log-logistic exponential degradation model and the skew-normal exponential degradation model is provided. In addition, the two models are applied to fatigue-crack data while considering convergence analysis of the MCMC chains. Additionally, the conclusion of Chapter 5 is provided.

Main findings, discussion of the findings in this thesis and some recommendations for future research are presented in Chapter 6.

CHAPTER II

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter presents an overview of some general degradation models, methods for modelling TTF distribution using methods of parameter estimation, particularly classical and Bayesian approach, and the computational procedures involved. In the reliability studies, degradation data are more commonly used since it is inexpensive and more readily accessible than the lifetime data which involve records of failure time of a particular product. Issues in reliability studies, such as degradation, have been extensively considered in various engineering disciplines, leading to the development of numerous methodologies for investigation of the degradation phenomena of specific products. Degradation phenomena, including wear and tear of machinery parts, have been identified as potential causes of accidents. Consequently, early warning signs of possible accidents due to degradation phenomena could be identified based on research works such as TTF modelling (Gebraeel & Pan 2008).

Modelling the degradation measurements of a product over time is complex because the degradation measure may be influenced by many underlying degradation processes. For example, battery capacity degrades over time due to a combination of factors like chemical reaction and mechanical stress. Various methods of parameter estimation such as maximum likelihood estimation, least squares estimation and Bayesian approach have been employed to estimate the parameters of TTF distribution for the degradation models. Meeker et al. (1999) have described several useful reliability models that account for degradation over time based on the physical failure mechanisms. Additionally, they presented models related to both degradation and failure. They utilized the maximum likelihood method to estimate the parameters of mixed effect accelerated degradation models. Furthermore, the cumulative distribution function (cdf) of the TTF distribution is investigated using four different methods: analytical expressions, numerical evaluation, Monte Carlo evaluation and estimation. Confidence intervals for the quantities of interest such as model parameters and standard error for estimation cdf of the TTF are determined based on bootstrap sampling.

Additionally, non-parametric methods have also been utilized to estimate the parameters of TTF distributions for the degradation models. Al-Haj Ebrahem et al. (2009) applied the non-parametric kernel density method to estimate the TTF distribution and its percentiles based on a linear degradation model. Eidous et al. (2017) estimated the TTF distribution and its percentiles for a simple linear degradation model using the double kernel method. Al-Momani et al. (2021) presented the variable scale kernel method to estimate the TTF distribution under a linear degradation model. They have compared the performance of the variable scale kernel method with particular existing parametric methods, such as maximum likelihood method and ordinary least squares method, based on the assumption of various distributions of the degradation parameter including Weibull, exponential, half normal and log-logistic. The simulation study demonstrates that non-parametric methods could outperform parametric methods like maximum likelihood and ordinary least squares when the assumption of the data distribution is violated.

2.2 LITERATURE REVIEW ON GENERAL DEGRADATION MODEL

Degradation is a natural phenomenon observable in many different aspects of nature. In the context of reliability, degradation refers to a type of failure that occurs over time, such as the wear and tear of machine parts. This degradation can render machine parts non-functional, necessitating their replacement. The data obtained from the degradation process, known as default data, are cheaper and easier to collect as compared to the failure data. Consequently, for assessing the reliability of certain products, degradation models are applied to degradation data rather than failure data, which are rare and costly to obtain.

Degradation modelling is an effective approach for reliability assessment, predicting remaining useful life, maintenance planning and prognostic health management (Shahraki et al. 2017). Hamada (2005) illustrated the advantages of assessing reliability based on the laser degradation data by showing how Bayesian method provides a natural approach to analysing and modelling degradation data versus lifetime data. A linear degradation model with no intercept, assuming the degradation parameter follows a Weibull distribution, is considered for estimating the parameters of the TTF distribution. The Bayesian approach is applied to estimate the parameters of the Weibull distribution and the variance of the error terms for the linear degradation model. In this approach, flat priors are assumed for the parameters of interest. A comparison is made between the use of degradation data, lifetime data, and pseudo lifetime data in computing the reliability function and percentiles. It is concluded that the degradation data offers more advantages over lifetime data in reliability assessment.

Ba Dakhn et al. (2017b) applied the semi-parametric method to estimate the TTF distribution and its percentiles for a simple linear degradation model with no intercept. In this method, the parametric estimator is assumed to follow either half-normal or exponential distributions. The performance of this estimator is compared with the maximum likelihood estimator and ordinary least squares estimator. It was found that the performance of the semi-parametric method is superior when the distribution of the random effect is unknown.

Degradation models are crucial for monitoring equipment reliability because they enable predictive maintenance and optimize maintenance schedules that lead to less frequent maintenance and lower costs. These models are classified into two main methods: model-based and data-driven (Wu et al. 2019b). A commonly applied type of model-based degradation model is the general degradation path model (Ye and Xie 2015). This model uses mathematical representation to depict failure through degradation path models. Most of these models incorporate various mathematical functions, including linear, exponential, power and logarithm functions.

The general path degradation model is expressed as:

$$y_{ij} = D(t_{ij}\,; \varphi, \beta_i) + \varepsilon_{ij} \;, i = 1, \dots, n, j = 1, \dots, m_i \qquad \dots(2.1)$$

where y_{ij} denotes the observed degradation measurements of i^{th} unit at a time t_{ij}, the actual path of i^{th} unit at a time t_{ij} is $D(t_{ij}\,; \varphi, \beta_i)$, the term φ is a fixed-effect parameter, the degradation parameter for i^{th} unit is β_i, which is degradation rate, $\varepsilon_{ij} \overset{iid}{\sim} N(0, \sigma_\varepsilon^2)$ is a random error term where σ_ε^2 is a constant, n is the number of units that are tested, and m_i is the total number of observations on i^{th} unit. Here, $\{\varepsilon_{ij}\}$ and $\{\beta_i\}$ are assumed independent and $\beta_i{}'s$ are independent.

For the purpose of simplifying the explanation, all the subscripts are dropped. The actual path of the general degradation model for the critical degradation level, denoted as D_f, as described in Equation (2.1), is written as:

$$D_f = D(T\,; \varphi, \beta) \qquad \dots(2.2)$$

Equation (2.2) can be expressed in terms of several mathematical functions such as linear, power and exponential. These forms are utilized to estimate TTF distribution and its percentiles.

2.2.1 Linear Degradation Model

A linear model depicts a linear correlation between two variables, such as years of experience with income, length and weight of individuals, and the level of product degradation with time.

Previous studies such as Al-Haj Ebrahem et al. (2009a) has examined the linear degradation path model involving the slope β given by:

$$D_f = \varphi + \beta T \qquad \dots(2.3)$$

While the other studies such as Eidous et al. (2017) and Shat and Schwabe (2024) have considered the linear degradation model with no intercept, which means $\varphi = 0$; thus:

$$D_f = \beta T \qquad \ldots(2.4)$$

The linear degradation model is commonly applied in the field of engineering reliability since it is the simplest model to work with in the case of estimating TTF distribution (Rawashdeh et al. 2018b).

2.2.2 Exponential Degradation Model

Several researchers apply nonlinear degradation path models to estimate the TTF distribution. For instance, Yang et al. (2021) have proposed a generic prognostic framework with health index (HI) to predict the remaining useful life of induction motors. They have used exponential degradation model for describing the critical degradation level at time *t* denoted as HI(*t*). This model is expressed as the following:

$$HI(t) = \varphi e^{\beta t} + \varphi_o \qquad \ldots(2.5)$$

where φ and φ_o are fixed effect parameters.

Zhang et al. (2018) have assumed the exponential degradation model for predicting RUL for lithium-ion batteries. Liu et al. (2021) have described the general drift in the degradation process by three forms of degradation models, which are linear, power and exponential, to predict the RUL for the rolling bearings.

2.2.3 Power Degradation Model

A second nonlinear general degradation model based on Equation (2.2) is the power degradation model, which is another mathematical approach used to describe the degradation behaviour of components or systems over time. Siju and Kumar (2018) have considered the path degradation model which comprises of a power and exponential functions for modelling TTF distribution. They have applied the exponential and the power degradation models to estimate the TTF and its percentiles where the degradation parameter β follows an exponential distribution. They expressed the exponential and the power degradation path models respectively as the following:

$$y = \varphi\, e^{\beta t}, \varphi, \beta > 0, t \geq 0 \quad \ldots(2.6)$$

$$y = \varphi\, t^{\beta}, \varphi, \beta > 0, t \geq 0 \quad \ldots(2.7)$$

Sindhu and Atangana (2021) have used inverse power law for reliability analysis of lifespan of electronic equipment based on exponential inverse Weibull distribution. González et al. (2011) have applied the power degradation model for assessing the reliability of concentrator photovoltaic modules operating outdoors in real-time conditions.

2.3 MODELING TIME-TO-FAILURE DISTRIBUTION

Statisticians employ the general degradation path model to estimate the TTF distribution and its percentiles. In this model, the degradation parameter follows either a finite combination of two or more probability distributions or a specific probability distribution (Sen et al. 2016). In modelling lifetime data with the general degradation path model, various probability distributions are commonly applied to describe the distribution of the degradation parameter, such as log-logistic distribution Akhtar and Khan (2014) and Dandis et al. (2012), exponential distribution Abong et al. (2017) and Siju and Kumar (2018), xgamma distribution Yadav et al. (2021), half-normal distribution Ba Dakhn et al. (2017) or Weibull distribution Ababneh and Ebrahem (2018) and Yu (2006).

2.3.1 Skew-Normal Distribution

Flexible distributions known as skew-symmetric distributions have been identified as suitable for modelling unconventional characteristics in the degradation data, such as high skewness and multimodality (Alhamidie et al. 2019 & Ghaderinezhad et al. 2020). The skew-normal distribution can vary from symmetric to asymmetric, based on its skewness parameter values, leading to normal distribution if skewness parameter is zero and half-normal as the skewness parameter value approaches infinity (Pan et al. 2018). Ba Dakhn et al. (2023) found that the skew-normal distribution offers flexibility in degradation modelling, particularly in determining posterior distributions of linear degradation model. They have shown that assuming the degradation parameter following the skew-normal distribution contributes to an improved estimation of the parameters of the TTF distribution.

The flexibility of the skew-normal distribution makes it quite versatile in the field of degradation modelling. In addition, Tsai and Lin (2015) have assumed that the error term in the nonlinear accelerated destructive degradation test model follow the skew-normal distribution. To estimate the lifetime distribution, Pan et al. (2018) have assumed that the drift parameter in the Wiener degradation model follows a skew-normal distribution. Chen et al. (2019) have presented the stochastic degradation model based on the inverse Gaussian process where the degradation rate parameter is assumed to follow a skew-normal distribution.

A random variable x follows a skew-normal distribution if its probability density function (pdf) is given by:

$$g_X(x) = 2\,\phi(x)\,\Phi(\lambda x)\ ,x \in \mathbb{R} \qquad \ldots(2.8)$$

where $\phi(.)$ is the pdf and $\Phi(.)$ is the cdf of the standard normal distribution respectively, and $\lambda \in \mathbb{R}$ is the skewness or shape parameter.

According to Azzalini and Capitanio (2014), in the case where β in Equation (2.2) follows a skew-normal distribution, which is a flexible distribution incorporating a skewness parameter λ as well as location parameter μ and scale parameter σ, denoted as $\beta \sim SN(\mu, \sigma^2, \lambda)$, the cdf of β based on the skew-normal distribution is given as:

$$G_{\beta-SN}(\beta;\ \mu, \sigma^2, \lambda) = \Phi\left(\frac{\beta-\mu}{\sigma}\right) - 2T\left(\frac{\beta-\mu}{\sigma}, \lambda\right) \qquad \ldots(2.9)$$

and the pdf of β based on the skew-normal distribution is given as:

$$g_{\beta-SN}(\beta;\ \mu, \sigma^2, \lambda) = \frac{2}{\sigma}\,\phi\left(\frac{\beta-\mu}{\sigma}\right)\Phi\left(\lambda\left(\frac{\beta-\mu}{\sigma}\right)\right) \qquad \ldots(2.10)$$

where $\beta,\ \mu \in \mathbb{R}, \sigma^2 \in \mathbb{R}^+$ and $\boldsymbol{T}$ *(h, a)* is Owen's $\boldsymbol{T}$ function defined as:

$$\boldsymbol{T}(h,a) = \frac{1}{2\pi}\int_0^a \frac{e^{-\frac{h^2(1+x^2)}{2}}}{1+x^2}\,dx\,, -\infty < a, h < +\infty \qquad \text{…(2.11)}$$

The equations described above are utilized in this research work for modelling TTF distribution based on various general degradation models and will be explained in details in the subsequent chapters of this thesis.

Recently, some authors have explored skewed and heavy-tailed distributions to describe the slope parameter of the degradation model. For instance, Oliveira et al. (2018) have assumed the slope parameter follows a scale and log-scale mixture of the skew-normal distribution to account for skewness and heavy-tailed behaviour observed in the data. They applied this model to analyse train wheel degradation data and found that it outperformed the Weibull degradation model due to the presence of the heavy tail properties in the data. The skew-normal linear degradation model thus presents a viable alternative for analysing degradation data. Consequently, this study further investigates the performance of the skew-normal linear degradation model, comparing it with a log-logistic linear degradation model through simulation studies and real-world applications using GaAs laser degradation data.

2.3.2 Log-logistic Distribution

Let β be a log-logistic random variable in Equation (2.2). Then, the cdf and the pdf of β in the second case according to Muse et al. (2021) are given respectively as follows:

$$G_{\beta-LL}(\beta) = \frac{1}{1+\left(\frac{\beta}{\alpha}\right)^{-\omega}} \qquad \text{…(2.12)}$$

and

$$g_{\beta-LL}(\beta) = \frac{\left(\frac{\omega}{\alpha}\right)\left(\frac{\beta}{\alpha}\right)^{\omega-1}}{\left(1+\left(\frac{\beta}{\alpha}\right)^{\omega}\right)^2} \qquad \text{…(2.13)}$$

where α and ω are the scale and the shape parameters of log-logistic distribution respectively.

The log-logistic distribution plays an important role in the analysis of reliability data (Akhtar & Khan 2014; Rachid & Naima 2021 & Yu & Wang 2024). It is commonly employed in degradation models to estimate TTF distribution. For instance, Rawashdeh et al. (2018) estimated the TTF distribution based on a linear degradation path with a random degradation parameter following the log-logistic distribution.

2.4 BAYESIAN APPROACH

One of the most widely used methods in statistics, which integrates prior and experimental information to draw inferences about a population of interest, is the Bayesian method. Over the past two decades, the Bayesian method has gained widespread acceptance as a solution to applied statistical problems in various fields such as econometrics, education, engineering, and medicine. Furthermore, this method has been extensively utilized in literature, facilitated by the discovery of the MCMC method which is a computational tool for handling complex posterior distributions.

The Bayesian method combines information from two sources of information, which are experimental data and prior knowledge to make inferences about the parameters of interest, say $\boldsymbol{\theta}$. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_i)$ and assume that θ*'s* are independent. Let the probability distribution of $\boldsymbol{\theta}$ be denoted as $q(\boldsymbol{\theta})$. If t is a random observation having the pdf $f(t|\boldsymbol{\theta})$, then the joint probability distribution of t and $\boldsymbol{\theta}$ is given by

$$f(t, \boldsymbol{\theta}) = f(t|\boldsymbol{\theta})q(\boldsymbol{\theta}) \quad \ldots(2.14)$$

and the marginal distribution of *t* is:

$$m(t) = \int f(t|\boldsymbol{\theta})q(\boldsymbol{\theta})d\boldsymbol{\theta} \quad \ldots(2.15)$$

Let us consider a random sample of size n, denoted $\boldsymbol{t}$. By combining the sample information contained in $f(\boldsymbol{t}|\boldsymbol{\theta})$ and the joint prior distribution $q(\boldsymbol{\theta}) = q_1(\theta_1) \ldots q_i(\theta_i)$, the posterior distribution $\pi(\boldsymbol{\theta}|\boldsymbol{t})$ is given by:

$$\pi(\boldsymbol{\theta}|\boldsymbol{t}) = \frac{f(\boldsymbol{t}|\boldsymbol{\theta})q(\boldsymbol{\theta})}{m(\boldsymbol{t})} \qquad \ldots(2.16)$$

Using the Bayesian mechanism involving the likelihood function and prior distributions, $\pi(\boldsymbol{\theta}|\boldsymbol{t})$ can be given by:

$$\pi(\boldsymbol{\theta}|\boldsymbol{t}) = \frac{L(\boldsymbol{\theta};\boldsymbol{t})\, q_1(\theta_1) \ldots q_i(\theta_i)}{\int \ldots \int L(\boldsymbol{\theta};\boldsymbol{t})\, q_1(\theta_1) \ldots q_i(\theta_i) d\theta_1 \ldots d\theta_i} \qquad \ldots(2.17)$$

where $L(\boldsymbol{\theta};\boldsymbol{t})$ is the likelihood function for the parameters $\boldsymbol{\theta}$ and $q_i(\theta_i)$ is the prior distribution of θ_i.

2.4.1 Some Brief Outline of the Bayesian Method Applied in Degradation Models

The Bayesian method is widely used in the degradation models by many researchers. For instance, Al-Haj Ebrahem et al. (2009a) fitted two types of data, which are grouped and non-grouped, using a linear mixed degradation model based on the Bayesian approach. In this approach, when estimating the parameters for the failure time distribution, the random slope parameter is assumed to follow an exponential distribution with mean μ. The prior for μ is assumed to be inverse gamma, while the intercept parameter is assumed to be uniform. The Bayesian approach was found to be more efficient for non-grouped data when compared to grouped data.

Freitas et al. (2010) applied four methods of data analysis, namely the approximate method, the analytical method, the two-stage method and the numerical method, for determining the TTF distribution. In the Bayesian approach, the degradation rate is assumed to follow either a Weibull or a log-normal, while the hyperpriors are assumed weakly informative. They found that the results based on these two distributions of the degradation parameter were quite similar.

Rawashdeh et al. (2018) applied the Bayesian approach using the differential evolution Markov chain method to estimate parameters of the TTF distribution for grouped and non-grouped data. In the Bayesian approach, when estimating the parameters for the failure time distribution, the random degradation parameter is assumed to follow a log-logistic distribution with shape and scale parameters. Jeffery's prior is considered for shape and scale parameters while the intercept parameter is assumed uniform. The Bayesian approach is found to be more efficient for non-grouped data when compared to the grouped data.

2.4.2 Choice of the Prior Distributions

The Bayesian approach is widely used to estimate the parameters of the TTF distribution. Several studies, including those by Cao et al. (2018), Chen et al. (2020) and Zhao et al. (2021), have applied the Bayesian approach, incorporating both informative and non-informative prior distributions, to estimate the parameters and the percentiles of the TTF distribution. Prior sensitivity analyses involving informative, weakly informative and non-informative priors is considered in these works to determine which priors yield more precise results for the estimated parameters. The sensitivity of the choice of prior distribution in estimating the parameters and the percentiles of the TTF distribution has been extensively studied, with many researchers evaluating the impact of informative, weakly informative, and non-informative prior distributions (Fúquene Patiño et al. 2018 & Lemoine 2019). Since prior sensitivity analysis is important in Bayesian approach, in this study, the Bayesian approach which incorporate informative, weakly informative and non-informative priors, is applied to estimate the TTF distribution based on various general degradation path models.

2.5 COMPUTATION

Since it is often found that the joint posterior distribution of the parameters cannot be determined in a closed form, the MCMC method is applied (Fernandes et al. 2022). Specifically, samples from the joint posterior distributions can also be generated using the just another Gibbs sampler (JAGS) algorithm. In this study, the MCMC method in the JAGS platform is employed to estimate the parameters of the TTF distribution and its percentiles for both models.

The JAGS algorithm, developed by Martyn Plummer (Plummer 2003), facilitates MCMC simulation based on Gibbs sampling, making the posterior analysis comparatively easier. For details on the implementation of the JAGS algorithm, refer to (Coro 2017, Albert 2008 & Su & Yajima 2024). For computation, the programming language R, version 4.0.3, is used (R Core Team 2023). JAGS algorithm is flexible and can be easily implemented after having successfully identified both the likelihood function and the prior distributions.

2.6 MAIN SUMMARY FROM THE LITERATURE REVIEW

Although there are substantial works on degradation modelling, there are not much works on Bayesian degradation modelling involving skew-normal or log-logistic degradation parameter. Accordingly, a short summary of works on degradation modelling which are available in the literature considered to be most relevant in this work is given in Table 2.1.

Table 2.1 Summary of the works on degradation modelling in the literature review

Model	Name of authors and year	Method of analysis	Findings
	Ba Dakhn et al. (2017b)	Estimate the parameters and the percentiles of the TTF distribution with the degradation parameter follows either half normal or exponential distributions. This parametric estimator compared with the maximum likelihood and ordinary least squares estimators.	They found that the performance of the semi-parametric method is superior when the distribution of the degradation parameter is unknown.

Linear	Oliveira et al (2018)	They used the Bayesian approach under linear degradation model with degradation parameter assumed to follow scale and log-scale mixture of the skew-normal distributions to account for skewness and heavy-tailed behavior observed in the train wheel degradation data.	They found that the skew-normal degradation model outperformed the Weibull degradation model.
	Rawashdeh et al (2018)	They used Bayesian approach under linear degradation model with degradation parameter following either log-logistic or Weibull distributions	They found that the Bayesian approach is more efficient for non-grouped data when compared to the grouped data under log-logistic degradation parameter.
Power Exponential	Siju and Kumar (2018)	They used Bayesian approach under power and exponential general degradation models with degradation parameter following exponential distribution to estimate TTF distribution and its percentiles.	For increasing sample size, the maximum likelihood and Bayesian estimates of reliability approach the actual reliability.

CHAPTER III

BAYESIAN APPROACH FOR TTF DISTRIBUTION BASED ON LINEAR DEGRADATION MODEL

3.1 INTRODUCTION

Although there are many methods applied for estimating the parameters of TTF distribution and its percentiles based on the degradation models, Bayesian approach remains to be a viable alternative as compared to the frequentist approach. Modelling TTF distribution in this chapter is presented by using a linear degradation model where β is assumed either skew-normal or log-logistic distribution. The parameters of the TTF distribution under skew-normal degradation parameter are estimated under different prior assumptions of the parameters, which are informative, weakly informative and non-informative. Additionally, comparison of the performance of skew-normal and log-logistic linear degradation models is carried out using simulation and GaAs laser degradation data. The convergence diagnostic found using MCMC method is discussed.

3.2 TTF DISTRIBUTION FOR LINEAR DEGRADATION MODEL (TTFL)

In the linear degradation path model, the specific trajectory of the degradation model in Equation (2.2) of a particular unit can be expressed as a linear equation as follows:

$$D_f = \varphi + \frac{T}{\beta} \qquad \ldots(3.1)$$

The model includes parameters φ, which is degradation value when $T = 0$, β as a random effect parameter and T as a failure time. The slope in the linear equation is given as $\frac{1}{\beta}$. In contrast to Equation (2.3), this study modifies the linear degradation path model which is presented in Equation (3.1). Thus, the slope is $\frac{1}{\beta}$. This adjustment, which involves taking the reciprocal of the degradation parameter, is made for mathematical convenience. It is demonstrated in this chapter that this modification results in the pdf of the TTF distribution being skew-normal after having assumed β following skew-normal distribution, simplifying our data-generating task for ensuring identifiability. Based on Equation (3.1), the TTF is given by $T = \beta(D_f - \varphi)$.

Then, the cdf of TTFL, denoted by F_{T-L} is derived using the distribution function technique as follows:

$$\begin{aligned} F_{T-L}(t) &= P(T \leq t) \\ &= P\big(\beta(D_f - \varphi) \leq t\big) \\ &= P\left(\beta \leq \frac{t}{(D_f - \varphi)}\right) \\ &= G_\beta\left(\frac{t}{D_f - \varphi}\right), \quad t > 0 \end{aligned} \qquad \ldots(3.2)$$

where $G_\beta(.)$ is the cdf of β. By taking the derivative of the function G_β with respect to *t,* the pdf of the TTFL, denoted as f_{T-L}, is found as the following:

$$f_{T-L}(t) = \frac{1}{D_f - \varphi} g_\beta\left(\frac{t}{D_f - \varphi}\right) \qquad \ldots(3.3)$$

As noted in Equations (3.2) and (3.3), the estimation of the cdf and pdf of the TTF distribution is dependent on the distribution of the degradation parameter β which is assumed to be known. The cdf and pdf of the TTFL are presented in two cases of the distribution for the degradation parameter which are skew-normal and log-logistic distributions.

3.2.1 Linear Degradation Model with Skew-Normal Degradation Parameter

Assume that parameter β in Equation (3.2) follows the skew-normal distribution with cdf and pdf given by Equation (2.9) and Equation (2.10) respectively. Based on

Equation (2.9), which is the definition of the skew-normal random variable, Equation (3.2) is modified to obtain the cdf of TTFL, denoted by F_{T-LSN}, as given by:

$$F_{T-LSN}(t) = \Phi\left(\frac{\frac{t}{D_f-\varphi}-\mu}{\sigma}\right) - 2\boldsymbol{T}\left(\frac{\frac{t}{D_f-\varphi}-\mu}{\sigma}, \lambda\right) \qquad \ldots(3.4)$$

By modifying Equation (3.2) and using Equation (2.10), the pdf of TTFL with skew-normal degradation parameter, denoted by f_{T-LSN}, is given as follows:

$$f_{T-LSN}(t) = \frac{2}{(D_f-\varphi)\sigma}\phi\left(\frac{t-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right)\Phi\left(\lambda\left(\frac{t-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right)\right) \qquad \ldots(3.5)$$

Note that Equation (3.5) can also be obtained by taking the derivative of F_{T-LSN} with respect to t. See Appendix A for the derivation.

The cdf and the pdf of TTF in Equations (3.4) and (3.5) respectively are the cdf and the pdf of the skew-normal distribution with location parameter $(D_f - \varphi)\mu$, scale parameter $(D_f - \varphi)\sigma$ and shape parameter λ. To compute the 100 r^{th} percentile, denoted as t_{r-LSN}, we determine the inverse cdf of the Equation (3.2) and solved for t_{r-LSN} as follows:

Let

$$G_\beta\left(\frac{t_{r-LSN}}{(D_f-\varphi)}\right) = r \qquad \ldots(3.6)$$

By taking the inverse function for both sides of Equation (3.6), we get

$$\frac{t_{r-LSN}}{(D_f-\varphi)} = G_\beta^{-1}(r) \qquad \ldots(3.7)$$

Then,

$$t_{r-LSN} = \left(D_f - \varphi\right) G_{\beta}^{-1}(r) \qquad \ldots(3.8)$$

Equation (3.8) provides an expression on how to determine the r^{th} percentiles of TTFL under skew-normal degradation parameter, i.e. t_{r-LSN}, which is found by taking the product of the r^{th} percentile of the distribution function of β with $\left(D_f - \varphi\right)$.

Let $t_1, t_2, \ldots, t_n$ denote a random sample of size n from the TTF distribution with parameters λ, σ, μ and φ. Based on Equation (3.5), we obtain the likelihood of the samples $t_1, t_2, \ldots, t_n$ which is given by:

$$L(\mu, \sigma, \lambda, \varphi; \boldsymbol{t}) = \left(\frac{2}{(D_f-\varphi)\sigma}\right)^n \prod_{i=1}^{n} \phi\left(\frac{t_i-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right) \Phi\left(\lambda\left(\frac{t_i-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right)\right) \qquad \ldots(3.9)$$

Equation (3.9) is used to obtain the joint posterior distribution in Section 3.3.1.

3.2.2 Linear Degradation Model with Log-logistic Degradation Parameter

In this subsection, modelling of the TTF distribution and its percentiles based on the linear degradation model with degradation parameter β which follows the log-logistic distribution is provided. Consider the cdf and pdf of β which are given respectively in Equations (2.12) and (2.13). The cdf of the TTFL with log-logistic degradation parameter, denoted as F_{T-LL}, can be determined by modifying Equation (3.2) based on Equation (2.12) as:

$$F_{T-LL}(t; \alpha, \omega, \varphi) = \frac{1}{1+\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-\omega}} \qquad \ldots(3.10)$$

Then, the pdf of the TTFL with log-logistic degradation parameter, denoted as f_{T-LL}, is found by taking the derivative of both sides of Equation (3.10) with respect to t as the following:

$$\frac{d(F_{T-LL})}{dt} = \frac{d\left(\frac{1}{1+\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-\omega}}\right)}{dt}$$

$$= \frac{-\left(\frac{-\omega}{(D_f-\varphi)\alpha}\right)\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-\omega-1}}{\left(1+\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-\omega}\right)^2}$$

$$= \frac{\left(\frac{\omega}{(D_f-\varphi)\alpha}\right)\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-\omega-1}}{\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{-2\omega}\left(1+\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{\omega}\right)^2}$$

Then,

$$f_{T-LL}(t;\alpha,\omega,\varphi) = \frac{\omega}{(D_f-\varphi)\alpha}\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{\omega-1}\left(1+\left(\frac{t}{(D_f-\varphi)\alpha}\right)^{\omega}\right)^{-2} \quad \ldots(3.11)$$

Note that the TTF distribution in Equations (3.10) and (3.11) is also the log-logistic distribution with scale parameter $(D_f-\varphi)\alpha$ and the shape parameter ω. Based on solving Equation (3.10) for t, the 100 r^{th} percentiles of the TTFL with the log-logistic degradation parameter, denoted by t_{r-LL}, is determined as the following:

Let

$$\frac{1}{1+\left(\frac{t_{r-LL}}{(D_f-\varphi)\alpha}\right)^{-\omega}} = r$$

Then, we have:

$$\left(\frac{t_{r-LL}}{(D_f-\varphi)\alpha}\right)^{-\omega} = \frac{1-r}{r}$$

$$(t_{r-LL})^{\omega} = \left((D_f-\varphi)\alpha\right)^{\omega}\left(\frac{r}{1-r}\right)$$

Finally, t_{r-LL} is given as:

$$t_{r-LL} = (D_f - \varphi)\alpha \left(\frac{r}{1-r}\right)^{\frac{1}{\omega}} \qquad \ldots(3.12)$$

Suppose that $t_1, t_2, \ldots, t_n$ is a random sample size n generated from the log-logistic distribution in Equation (3.11), the likelihood function of the parameters of the TTFL is given by:

$$L(\alpha, \omega, \varphi; \boldsymbol{t}) = \left(\frac{\omega}{(D_f-\varphi)\alpha}\right)^n \prod_{i=1}^{n} \left(\frac{t_i}{(D_f-\varphi)\alpha}\right)^{\omega-1} \left(1 + \left(\frac{t_i}{(D_f-\varphi)\alpha}\right)^{\omega}\right)^{-2} \qquad \ldots(3.13)$$

This likelihood function is applied in the Bayesian approach given in Section 3.3.2.

3.3 BAYESIAN MODELING OF TTFL

The Bayesian method which involves informative, weakly informative and non-informative priors is considered to estimate the parameters and the percentiles of the TTF distribution. The parameters of the TTF distribution and its percentiles are determined based on the linear degradation model where the degradation parameter is assumed to follow either the skew-normal or the log-logistic distributions. The basic mechanism in the Bayesian approach involves updating the prior distribution for the parameters of interest using the current information based on the following relationship:

$$\text{posterior} \propto \text{prior} \times \text{likelihood} \qquad \ldots(3.14)$$

Based on the relationship (3.14), the joint posterior density function is determined based on the different assumptions on the distribution of the degradation parameter. In particular, the degradation parameter is assumed skew-normal.

3.3.1 Posterior Density of TTFL with Skew-Normal Degradation Parameter

In this subsection, the Bayesian approach is considered for the skew-normal linear degradation model. Both informative and non-informative priors are represented by gamma and uniform distributions respectively with certain known hyperparameters. It is known that if non-informative priors are used to specify the prior distribution, the sample data would dominate the posterior distribution, particularly when the sample size is large. Based on Equation (2.17), the posterior distribution of the TTFL with skew-normal degradation parameter, denoted as $\pi_{LSN}(\lambda,\sigma,\mu,\varphi|\boldsymbol{t})$, can be given by:

$$\pi_{LSN}(\lambda,\sigma,\mu,\varphi|\boldsymbol{t}) = \frac{L(\lambda,\sigma,\mu,\varphi;\boldsymbol{t})\; q_1(\lambda)q_2(\sigma)q_3(\mu)q_4(\varphi)}{\int\int\int\int L(\lambda,\sigma,\mu,\varphi;\boldsymbol{t})\; q_1(\lambda)q_2(\sigma)q_3(\mu)q_4(\varphi)d\lambda d\sigma d\mu d\varphi} \quad \ldots(3.15)$$

where $L(\lambda,\sigma,\mu,\varphi;\boldsymbol{t})$ is the likelihood function of the parameters λ,σ,μ and φ of the TTF distribution, and q_1, q_2, q_3 and q_4 are the marginal prior distributions of λ,σ,μ and φ respectively.

For the prior distributions, gamma distribution is the usual prior distribution used in estimating parameters of TTF distribution due to its positive domain and uniform distribution is usually applied as a non-informative prior distribution. Then, the parameters σ,μ and φ of the TTF distribution are assumed to follow the uniform distributions while λ is assumed to follow the gamma distribution. The gamma priors are defined as either informative or weakly informative prior depending on the values of the hyperparameters which is illustrated later in the simulation study section. The performance of the method under the different prior assumptions is compared using a simulation study based on the MCMC method as well as a real data application. The prior distributions considered are given as follows:

i) Given that $\lambda \sim Gamma(a,b)$, then $q_1(\lambda) = \begin{cases} \frac{\lambda^{a-1} e^{\frac{-\lambda}{b}}}{\Gamma(a)b^a} & , \lambda > 0 \\ 0 & ,\text{otherwise} \end{cases}$

ii) Given that $\sigma \sim Uniform(0,c)$, then $q_2(\sigma) = \begin{cases} \frac{1}{c} & , 0 < \sigma < c \\ 0 & ,\text{otherwise} \end{cases}$

iii) Given that $\mu \sim Uniform(0,d)$, then $q_3(\mu) = \begin{cases} \frac{1}{d} & , 0 < \mu < d \\ 0 & ,\text{otherwise} \end{cases}$

iv) Given that $\varphi \sim Uniform(0,k)$, then $q_4(\varphi) = \begin{cases} \frac{1}{k} & , 0 < \varphi < k \\ 0 & ,\text{otherwise} \end{cases}$

where *a*, *b, c, d* and *k* are constants.

Based on Equation (3.15), $\pi_{LSN}(\lambda, \sigma, \mu, \varphi|\boldsymbol{t})$ is proportional to:

$$\frac{\lambda^{a-1}\, e^{\frac{-\lambda}{b}}}{\Gamma(a)\; c\; d\; k\; b^{a}}\left(\frac{2}{(D_f-\varphi)\sigma}\right)^{n}\prod_{i=1}^{n}\phi\left(\frac{t_i-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right)\Phi\left(\lambda\left(\frac{t_i-(D_f-\varphi)\mu}{(D_f-\varphi)\,\sigma}\right)\right) \quad \ldots(3.16)$$

To estimate the parameters of TTFL, say λ for example, the marginal posterior density of λ given $\boldsymbol{t}$ is to be determined by integration as follows:

$$f(\lambda|\boldsymbol{t}) \propto \iiint \pi_{LSN}(\lambda, \sigma, \mu, \varphi|\boldsymbol{t})\; d\sigma\; d\mu\; d\varphi \quad \ldots(3.17)$$

The Equation (3.17) is rather complicated since it involves a high dimensional integration with a complex integrand. Thus, the parameters λ, σ, μ and φ in Equation (3.16) are estimated using MCMC method based on the R programming carried out under JAGS platform.

3.3.2 Posterior Density of TTFL with Log-logistic Degradation Parameter

As in Section 3.3.1, the Bayesian approach is again applied. Now, it is applied on the log-logistic linear degradation model for estimating the parameters of the TTF distribution and its percentiles. In order to compare the performance of the skew-normal and the log-logistic linear degradation models under the Bayesian approach, the choice of the prior distribution for both models have to be made comparable. The prior distributions for the scale and fixed-effect parameters for each model are assumed to follow uniform distribution as

i) $\alpha \sim Uniform(0, c)$. Then, $q_1(\alpha) = \begin{cases} \frac{1}{c} & , \ 0 < \alpha < c \\ 0 & , \text{otherwise} \end{cases}$

ii) $\varphi \sim Uniform(0, k)$. Then, $q_3(\varphi) = \begin{cases} \frac{1}{k} & , \ 0 < \varphi < \ k \\ 0 & , \text{otherwise} \end{cases}$

iii) $\omega \sim Gamma(a,b)$. Then, $q_2(\omega) = \begin{cases} \frac{\omega^{a-1} e^{\frac{-\omega}{b}}}{\Gamma(a) b^a} & , \ \omega > 0 \\ 0 & , \text{otherwise} \end{cases}$

Note that the location parameter, which is another parameter for the skew-normal distribution, is assumed fixed and it is not estimated. In addition, for both models, the initial and the critical degradation values are also assumed to be the same. Then, it can be shown that the joint posterior distribution, denoted as $\pi_{LL}(\alpha, \omega, \varphi; \boldsymbol{t})$ is proportional to:

$$\frac{\omega^{a-1} e^{\frac{-\omega}{b}}}{\Gamma(a)\ c\ k\ b^a} \left(\frac{\omega}{(D_f - \varphi)\alpha}\right)^n \prod_{i=1}^{n} \left(\frac{t_i}{(D_f - \varphi)\alpha}\right)^{\omega - 1} \left(1 + \left(\frac{t_i}{(D_f - \varphi)\alpha}\right)^{\omega}\right)^{-2} \quad \ldots(3.18)$$

The marginal posterior distributions of the parameters for the TTF distribution, such as $\pi(\alpha|\boldsymbol{t})$, $\pi(\omega|\boldsymbol{t})$ and $\pi(\varphi|\boldsymbol{t})$ are also quite complicated. Thus, estimation of the parameters α, ω and φ is carried out based on the MCMC method run under R program in the JAGS platform.

3.4 SIMULATION STUDY OF TTFL

In this study, the MCMC method in the JAGS platform is implemented to estimate the parameters of the TTF distribution and its percentiles for both models. The performance of Bayesian model under informative, weakly informative and non-informative priors is studied based on simulated data which are generated using certain true values of the parameter and the critical level of degradation D_f. The assumed true parameter values are given as follows: $\lambda = 3$; $\sigma = 2$; $\mu = 1$. Additionally, the initial value of fixed-effect degradation parameter φ and D_f are respectively given by 6 and 20 since those values have been applied in the studies by Rawashdeh et al. (2018) and Ba Dakhn et al. (2017). The sample sizes which are considered in this simulation study involves n = 30, 60 and 200. The choice of these sample sizes reflects the gradual change of the sample size from small to large size.

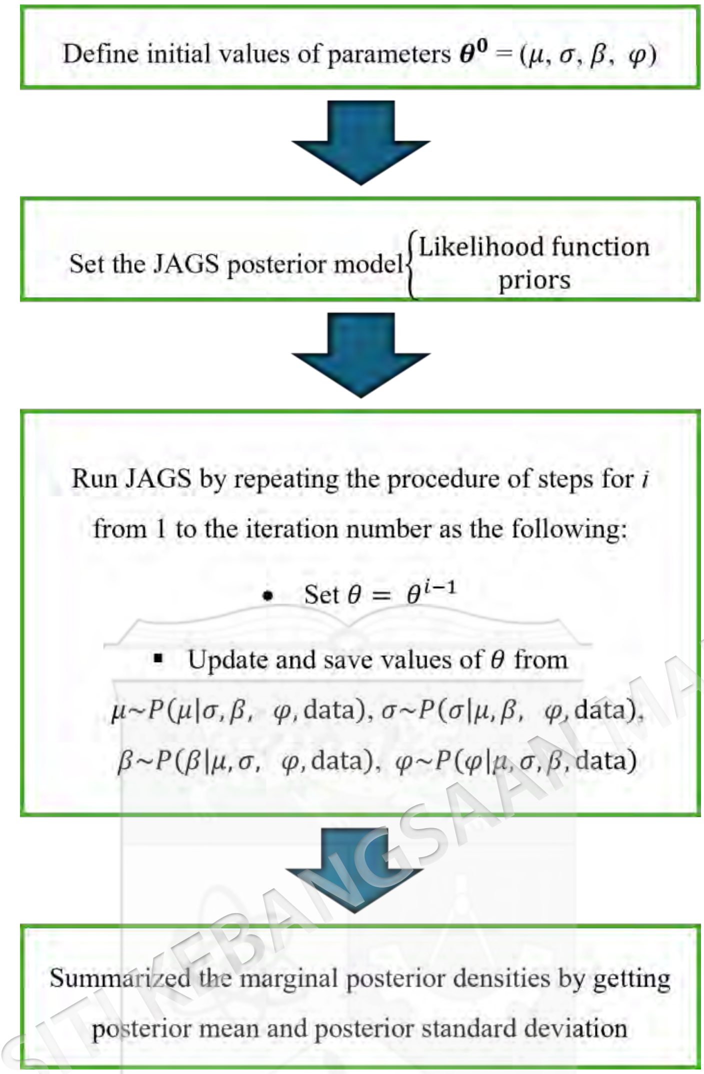

Figure 3.1 Flowchart for determining the marginal posterior distribution of the parameters of TTF distribution using JAGS

Based on these assumed values, the data for TTF distribution are generated using the skew-normal distribution with location parameter $(D_f - \varphi)\mu$, scale parameter $(D_f - \varphi)\sigma$ and shape parameter λ. Additionally, the r^{th} percentiles of TTF distribution are assumed to be 0.05, 0.2, 0.5, 0.75 and 0.9. Since determination of the marginal posterior distribution of the parameters involving some complex integrations, MCMC method is applied involving 100000 iterations where 50000 is used as burn-in. In particular, samples from the joint posterior distribution of the TTFL in Equation (3.5) are generated using the JAGS algorithm. For further illustration of the simulation process and determination of the marginal posterior densities for the linear degradation model, the flowchart in Figure 3.1 is provided.

Refer to Appendix B to see the R codes for estimation of the parameters and percentiles of TTFL.

3.4.1 Bayesian Analysis for Skew-Normal Linear Degradation Model Based on Several Different Priors

In this subsection, various types of priors are presented, involving informative, weakly informative and non-informative priors. The weakly informative gamma prior is represented based on the assumed values of the shape and scale hyperparameters of 0.1 and 100 respectively while the informative gamma prior is represented by both hyperparameter values of 2. Additionally, the non-informative prior is presented by assuming all the parameters to follow the uniform distribution.

The results of the parameter estimation and determination of certain percentiles of the TTF based on Bayesian approach of the skew-normal linear degradation model under informative, weakly informative and non-informative priors for the different sample sizes are provided in Tables 3.1 to 3.3. The properties of the estimated parameters of the posterior distribution (PE) are evaluated using bias (B) and standard deviation (SD) values found based on the posterior parameter estimates determined using the JAGS platform. The mean of the estimated values of the parameter is considered as the estimate for the parameter mean. In the context of parameter estimation, estimators with small B and SD are preferred.

Table 3.1 PE, B and SD for the parameters and certain percentiles of the TTFL with skew-normal degradation parameter under weakly informative, informative and non-informative priors for $n = 30$

Parameters	**Weakly Informative prior**			**Informative prior**			**Non-informative prior**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	1.474	-0.475	0.481	1.227	-0.227	0.433	1.260	-0.260	0.422
$\sigma = 2$	1.579	0.422	0.813	2.165	-0.165	0.734	2.115	-0.115	0.757
$\lambda = 3$	3.189	-0.190	9.589	2.955	0.045	2.045	2.633	0.367	1.493
$\varphi = 6$	4.468	1.532	3.390	6.823	-0.823	3.278	6.680	-0.680	3.333
$t_{0.05}$=9.432	6.627	2.805	4.791	8.225	1.207	4.051	8.166	1.266	4.128
$t_{0.2}$=19.598	18.726	0.872	3.665	19.059	0.539	3.313	19.168	0.430	3.277
$t_{0.5}$=32.816	32.803	0.013	3.297	32.721	0.095	3.413	32.817	-0.001	3.366
$t_{0.75}$=46.209	45.272	0.937	3.750	45.924	0.285	4.227	45.897	0.312	4.186
$t_{0.9}$=60.056	57.203	2.853	5.516	59.221	0.835	5.907	59.016	1.040	5.859

Table 3.2 PE, B and SD for the parameters and certain percentiles of the TTFL with skew-normal degradation parameter under weakly informative, informative and non-informative priors for n = 60

Parameters	Weakly Informative prior			Informative prior			Non-informative prior		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	1.236	-0.236	0.397	1.181	-0.181	0.371	1.168	-0.168	0.367
$\sigma = 2$	2.417	-0.417	0.739	2.501	-0.501	0.678	2.518	-0.518	0.686
$\lambda = 3$	3.280	-0.280	1.713	3.491	-0.491	1.350	3.554	-0.554	1.164
$\varphi = 6$	7.123	-1.123	3.382	7.383	-1.383	3.222	7.380	-1.380	3.264
$t_{0.05}$= 9.432	9.028	0.405	3.214	9.512	-0.080	2.770	9.631	-0.199	2.734
$t_{0.2}$= 19.598	19.823	-0.225	5.405	19.912	-0.314	2.242	19.903	-0.305	2.200
$t_{0.5}$= 32.816	33.876	-1.160	2.434	33.879	-1.063	2.415	33.830	-1.015	2.412
$t_{0.75}$= 46.209	48.134	-1.925	3.139	48.146	-1.937	3.158	48.164	-1.955	3.191
$t_{0.9}$= 60.056	62.570	-2.514	4.518	62.056	-2.767	4.457	62.927	-2.871	4.494

Table 3.3 PE, B and SD for the parameters and certain percentiles of the TTFL with skew-normal degradation parameter under weakly informative, informative and non-informative priors for n = 200

Parameters	Weakly Informative prior			Informative prior			Non-informative prior		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	1.165	-0.165	0.326	1.150	-0.151	0.327	1.123	-0.123	0.320
$\sigma = 2$	2.567	-0.567	0.647	2.332	-0.332	0.611	2.328	-0.328	0.614
$\lambda = 3$	5.462	-2.462	6.890	3.334	-0.334	0.846	3.451	-0.451	0.870
$\varphi = 6$	7.977	-1.977	3.209	7.718	-1.718	3.316	7.617	-1.617	3.312
$t_{0.05}$=9.432	11.523	-2.090	1.194	9.296	0.136	1.340	9.347	0.085	1.327
$t_{0.2}$=19.598	19.906	-0.308	1.272	18.662	0.936	1.137	18.607	0.991	1.140
$t_{0.5}$=32.816	32.631	0.185	1.318	31.169	1.647	1.246	31.107	1.709	1.254
$t_{0.75}$=46.209	46.414	-0.205	1.682	43.989	2.221	1.572	44.004	2.205	1.587
$t_{0.9}$=60.056	60.719	-0.663	2.447	57.223	2.833	2.267	57.329	2.727	2.291

The estimators are evaluated based on the PE, B and SD. From Tables 3.1 to 3.3, we have found the following results:

i) The SD of the estimated values for the parameters μ, σ and λ and the percentiles for the TTF distribution found for the Bayesian approach of skew-normal linear degradation model under various priors decreases as n increases and this pattern is unclear for φ. For example, when n = 30, and in the case of non-informative prior SD = 3.333 while for n = 60 and 200 in the same case of non-informative prior, SD = 3.264 and 3.312 respectively.

ii) Based on the different types of priors, the SD of the r^{th} percentiles of the TTF distribution increases as r increases for all sample sizes except in the case of the low value of the r^{th} percentile. For example, when n = 30, and in the case of non-informative prior, SD = 4.128, 3.277, 3.366, 4.186 and 5.859 respectively for the smallest to the largest percentiles.

iii) In 10 out of 27 cases, B values of the estimated parameters and the percentiles under non-informative gamma prior are found to be smaller than those under weakly informative and informative gamma priors. For example, based on the non-informative prior for the parameter φ, in the case when n = 200, the B value $= -1.617$ while for the weakly informative and informative priors B = -1.977 and -1.718 respectively.

iv) For sample size of 60, the B of weakly informative is smaller. The results of B values for informative and non-informative priors are quite close for large sample size. For example, under the weakly informative prior for the parameter φ, the B value $= -1.123$ while for the informative and non-informative priors B = -1.383 and -1.380 respectively.

v) In 12 out of 27 cases from the SD values of the estimated parameters and the percentiles under informative gamma prior are smaller than those found under weakly informative and non-informative priors. These values are quite close for large sample size. For example, in the case of large sample size and 90th percentile, the SD values are 2.447, 2.267 and 2.291 under weakly informative, informative and non-informative priors, respectively.

Based on the above discussion we can say that, in general, the performance of the Bayesian approach of skew-normal linear degradation model under informative gamma prior is better than the performance of the Bayesian approach of skew-normal linear degradation model under weakly informative gamma prior and non-informative prior. In the case of large sample size, the results found based on the different prior assumptions are found to be quite close. This is not surprising since prior knowledge has lesser influence on the posterior as the sample size gets larger.

3.4.2 Comparison Between Log-logistic and Skew-Normal Degradation Parameters Under TTFL

In this subsection, the performance of Bayesian approach for modelling the TTF distribution and its percentiles based on linear degradation model with degradation parameter which follows log-logistic distribution and skew-normal distribution is investigated under informative gamma prior. Then, the comparison between these two models is presented without estimating the location parameter for skew-normal linear degradation model. All the true values of the parameters are assumed the same as given in Section 3.4.1. The results of the simulation study for comparison are provided in Tables 3.4 to 3.6.

Table 3.4 PE, B and SD of the parameters and the percentiles for the skew-normal and log-logistic linear degradation models for the simulated data for n = 30

Skew-normal linear degradation model				**Log-logistic linear degradation model**			
True parameters	**PE**	**B**	**SD**	**True parameters**	**PE**	**B**	**SD**
$\sigma = 2$	1.891	0.109	0.588	$\alpha = 2$	2.627	-0.627	0.666
$\lambda = 3$	4.459	-1.459	2.008	$\omega = 3$	3.148	-0.148	0.480
$\varphi = 6$	4.624	1.376	2.759	$\varphi = 6$	6.532	-0.532	3.332
$t_{0.05}$= 9.432	12.723	-3.291	3.208	$t_{0.05}$= 10.493	12.936	-2.442	2.230
$t_{0.2}$= 19.598	21.414	-1.816	2.445	$t_{0.2}$= 17.639	21.297	-3.658	2.547
$t_{0.5}$= 32.816	33.940	-1.125	2.864	$t_{0.5}$= 28.000	33.355	-5.355	3.361
$t_{0.75}$= 46.209	47.169	-0.960	4.231	$t_{0.75}$= 40.383	47.764	-7.381	5.660
$t_{0.9}$= 60.056	60.847	-0.791	6.014	$t_{0.9}$= 58.242	68.620	-10.378	10.892

Table 3.5 PE, B and SD of the parameters and the percentiles for the skew-normal and log-logistic linear degradation models for the simulated data for $n = 60$

Skew-normal linear degradation model				**Log-logistic linear degradation model**			
True parameters	**PE**	**B**	**SD**	**True parameters**	**PE**	**B**	**SD**
$\sigma = 2$	2.781	-0.781	0.676	$\alpha = 2$	2.324	-0.324	0.633
$\lambda = 3$	3.655	-0.655	1.335	$\omega = 3$	2.635	0.365	0.289
$\varphi = 6$	8.768	-2.768	2.405	$\varphi = 6$	6.992	-0.992	3.400
$t_{0.05}$= 9.432	7.040	2.392	2.583	$t_{0.05}$= 10.493	9.183	1.310	1.359
$t_{0.2}$= 19.598	17.246	2.352	1.761	$t_{0.2}$= 17.639	16.610	1.029	1.697
$t_{0.5}$= 32.816	31.130	1.685	2.196	$t_{0.5}$= 28.000	28.243	-0.243	2.399
$t_{0.75}$= 46.209	45.426	0.783	3.096	$t_{0.75}$= 40.383	43.122	-2.739	4.231
$t_{0.9}$= 60.056	60.151	-0.096	4.303	$t_{0.9}$= 58.242	65.985	-7.743	8.550

Table 3.6 PE, B and SD of the parameters and the percentiles for the skew-normal and log-logistic linear degradation models for the simulated data for n = 200

Skew-normal linear degradation model				Log-logistic linear degradation model			
True parameters	**PE**	**B**	**SD**	**True parameters**	**PE**	**B**	**SD**
$\sigma = 2$	2.136	-0.136	0.334	$\alpha = 2$	2.425	-0.425	0.636
$\lambda = 3$	3.244	-0.244	0.483	$\omega = 3$	3.399	-0.399	0.202
$\varphi = 6$	6.955	-0.955	3.450	$\varphi = 6$	7.072	-1.072	3.400
$t_{0.05}$= 9.432	8.262	1.170	1.635	$t_{0.05}$= 10.493	12.286	-1.793	0.773
$t_{0.2}$= 19.598	19.530	0.068	0.669	$t_{0.2}$= 17.639	19.442	-1.803	0.848
$t_{0.5}$= 32.816	32.954	-0.138	0.948	$t_{0.5}$= 28.000	29.266	-1.266	1.058
$t_{0.75}$= 46.209	45.482	0.727	1.519	$t_{0.75}$= 40.383	40.485	-0.102	1.661
$t_{0.9}$= 60.056	58.090	1.966	2.161	$t_{0.9}$= 58.242	58.242	2.215	2.973

From Tables 3.4 to 3.6, we have found the following results:

i) For most cases, the SD of the estimated percentiles in the Bayesian approach of skew-normal and log-logistic linear degradation models decreases as n increases. For example, $n = 30$, 60, 200 and the 75th percentiles, the SD values are 4.231, 3.096 and 1.519 respectively under skew-normal linear degradation model while under log-logistic linear degradation model the SD values for the same percentiles and sample sizes are 5.660, 4.231 and 1.661.

ii) The SD of the r^{th} percentile positions increase as r increase for all sample sizes, except in the case of the lower percentile for skew-normal linear degradation model. For example, for sample sizes $n = 60$, the SD values for the estimated percentiles from smallest assumed r to the largest value are 2.583, 1.761, 2.196, 3.096 and 4.303 respectively under skew-normal linear degradation model.

iii) In most cases for all sample sizes, the B values of the estimated percentiles in skew-norma linear degradation model decrease as r increase. For example, for sample size 60, the B values from the 5th to 90th percentiles are 2.392, 2.352, 1.685, 0.783 and -0.096 respectively.

iv) In 17 out of 24 cases, B values under skew-normal linear degradation model are smaller than the B values under log-logistic linear degradation model. For example, for different sample sizes and in the 75th percentiles, the B values are respectively -0.960, 0.783 and 0.727 under skew-normal linear degradation model while under log-logistic linear degradation model the B values for the same percentiles are -7.381, -2.739 and -0.102.

v) In 16 out of 24 cases, SD values for skew-normal linear degradation model are smaller than the SD values for log-logistic linear degradation model. For example, for $n = 30$, 60, 200 and in the 75th percentiles, the SD values are 4.231, 3.096 and 1.519 respectively under skew-normal linear degradation model while under log-logistic linear degradation model the SD values for the same percentiles are 5.660, 4.231 and 1.661 which indicate that each value of

SD under skew-normal linear degradation model is smaller than the corresponding value under log-logistic linear degradation model.

Based on the above notes, the performance of the Bayesian approach of the skew-normal linear degradation model is generally better than the performance of the Bayesian approach of log-logistic linear degradation model.

3.5 GaAs LASER DEGRADATION DATA

3.5.1 Describing Data

The Laser Degradation Data from Meeker and Escobar (1998) is considered. The dataset provides the percent increase in laser operating current for 15 GaAs laser devices which are tested at 80 ^{o}C when the output light is kept at a nearly constant reading. The data which are presented in Table 3.7 and Figure 3.1 consists of 15 units of device, and for each unit the measurements are taken at the time range from 250 to 4000 hours with step equals 250 hours. The failure is assumed to occur at the critical degradation level, which is denoted as D_f, and D_f is assumed equal to 5. Based on linear interpolation carried out on the data, we found one failure for each unit. The linear interpolation, for example for the degradation data of the first device, is done as follows:

$$\frac{D_f-4.91}{t_1-7} = \frac{5.48-D_f}{8-t_1} \qquad \ldots(3.19)$$

By substitute the value of D_f and doing the subtraction, we get

$$\frac{0.09}{t_1-7} = \frac{0.48}{8-t_1} \qquad \ldots(3.20)$$

Then,

$$t_1 = \frac{0.09\times8+7\times0.48}{0.09+0.48} \Rightarrow t_1 = \frac{0.72+3.36}{0.57} = 7.16 \qquad \ldots(3.21)$$

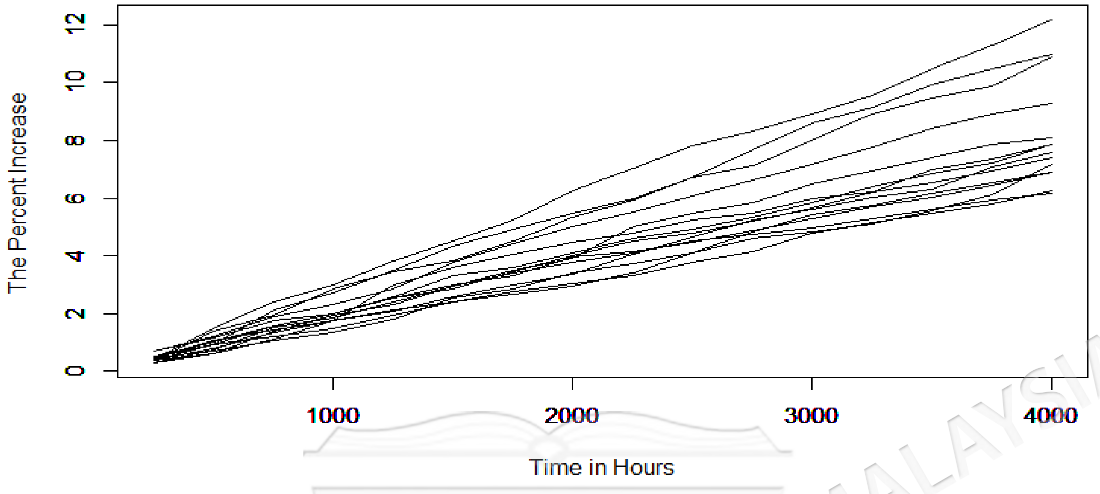

Figure 3.2 Percent Increase in Operating Current for GaAs Laser Tested at 80º C

Figure 3.2 shows that the laser degradation data follows a linear degradation path. Thus, it is reasonable to apply the linear degradation model.

Table 3.7 The data for GaAs Lasers Tested at 80 °C

Time (hours)	Device Unit Number														
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
250	0.47	0.71	0.71	0.36	0.27	0.36	0.36	0.46	0.51	0.41	0.44	0.39	0.30	0.44	0.51
500	0.93	1.22	1.17	0.62	0.61	1.39	0.92	1.07	0.93	1.49	1.00	0.80	0.74	0.70	0.83
750	2.11	1.90	1.73	1.36	1.11	1.95	1.21	1.42	1.57	2.38	1.57	1.35	1.52	1.05	1.29
1000	2.72	2.30	1.99	1.95	1.77	2.86	1.46	1.77	1.96	3.00	1.96	1.74	1.85	1.35	1.52
1250	3.51	2.87	2.53	2.30	2.06	3.46	1.93	2.11	2.59	3.84	2.51	2.98	2.39	1.80	1.91
1500	4.34	3.75	2.97	2.95	2.58	3.81	2.39	2.40	3.29	4.50	2.84	3.59	2.95	2.55	2.27
1750	4.91	4.42	3.30	3.39	2.99	4.53	2.68	2.78	3.61	5.25	3.47	4.03	3.51	2.83	2.78
2000	5.48	4.99	3.94	3.79	3.38	5.35	2.94	3.02	4.11	6.26	4.01	4.44	3.92	3.39	3.42
2250	5.99	5.51	4.16	4.11	4.05	5.92	3.42	3.29	4.60	7.05	4.51	4.79	5.03	3.72	3.78
2500	9.72	6.07	4.45	4.50	4.63	6.71	4.09	3.75	4.91	7.80	4.80	5.22	5.47	4.09	4.11
2750	7.13	6.64	4.89	4.72	5.24	7.70	4.58	4.16	5.34	8.32	5.20	5.48	5.84	4.83	4.38
3000	8.00	7.16	5.27	4.98	5.62	8.61	4.84	4.76	5.84	8.93	5.66	5.96	6.50	5.41	4.63
3250	8.92	7.78	5.69	5.28	6.04	9.15	5.11	5.16	6.40	9.55	6.20	6.23	6.94	5.76	5.38
3500	9.49	8.42	6.02	5.61	6.32	9.95	5.57	5.46	6.84	10.5	6.54	6.99	7.39	6.14	5.84
3750	9.87	8.91	6.45	5.95	7.10	10.5	6.11	5.81	7.20	11.3	6.96	7.37	7.85	6.51	6.16
4000	10.9	9.28	6.88	6.14	7.59	11.0	7.17	6.24	7.88	12.2	7.42	7.88	8.09	6.88	6.62

It appears that the percent increase in operating current increases linearly with respect to time, and the lines show zero intercept. The idea of considering the intercept value of 0, i. e. $\varphi = 0$, has been applied by Hamada (2005). Refer to Appendix C for the codes of R program.

3.5.2 Convergence Analysis under GaAs Laser Degradation Data

In this section, the GaAs laser degradation data is applied and the convergence of the series produced based on the MCMC simulation is investigated. The total number of the MCMC iterations involving the three chains is set to be 100000 where the first 50000 iterations are treated as burn-in. Following (Kundu 2008), the values of all hyperparameters are assumed equal to 2 for the uniform prior, while for the gamma prior distribution, the hyperparameter are set to have a variance of gamma distribution less than 1 as the case of informative prior which is provided in Section 3.5.3. The graphs of the trace plots, posterior density functions and autocorrelation of the parameters μ, σ and λ of the TTFL with skew-normal degradation parameter are determined and presented in Figure 3.3.

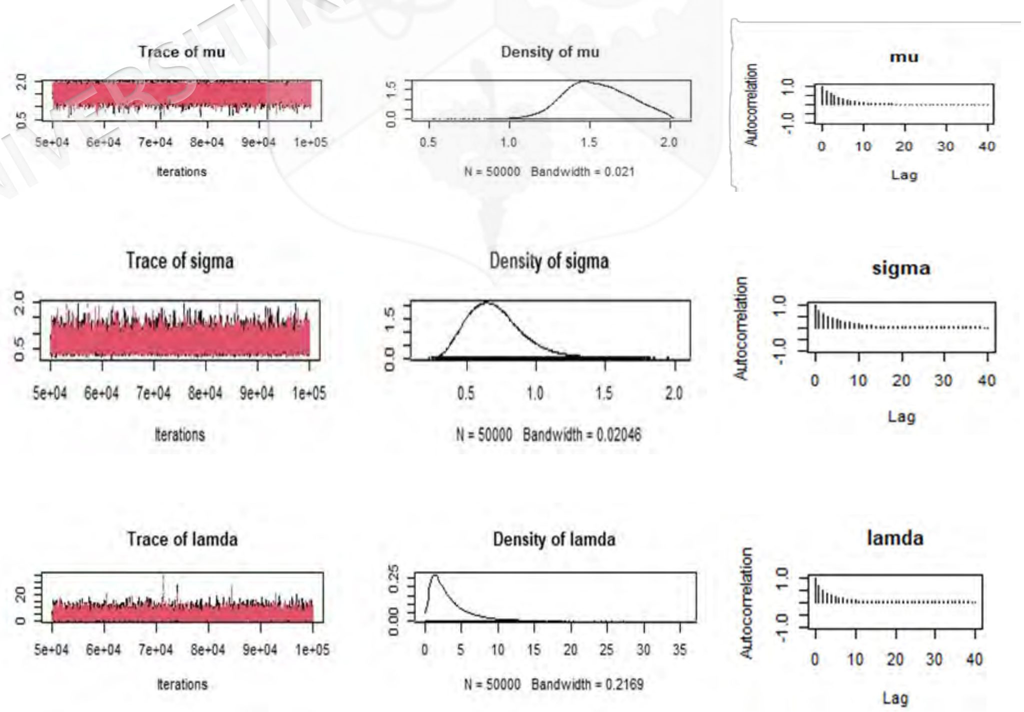

Figure 3.3 Trace plot, posterior density function and autocorrelation of the parameters μ, σ and λ based on the Bayesian analysis of GaAs Laser data under skew-normal linear degradation model

As shown in Figure 3.3, results found based on the proposed model converges satisfactorily well. Also, it appears that the autocorrelation graph with thinning of 1 reaches zero rather quickly, indicating that the chain is mixing adequately. In addition, the values of the potential scale reduction factor (psrf) of Gelman-Rubin and Geweke's stationarity test for convergence assessment which are provided in Table 3.8, are studied further in order to support the case of convergence.

Table 3.8 Geweke's stationarity test and the potential scale reduction factor to assess convergence of the MCMC chains for the parameters μ, σ and λ of the posterior distribution based on GaAs Laser data under skew-normal model

Parameters	**Geweke's stationarity test**			**Psrf**
	Chain 1	**Chain 2**	**Chain 3**	
μ	-1.613	-0.616	0.573	1
σ	0.518	1.010	0.551	1
λ	0.699	1.371	0.062	1

Generally, if the values of the Geweke's stationarity test are all between -1.96 and 1.96 and the values of psrf for all the parameters are found less than 1.1, the chain is said to converge to a stationary distribution (Brooks & Gelman 1998, Gelman & Rubin 1992 & Geweke 1991). It is apparent that the results from Table 3.8 satisfy the desired properties of convergence as outlined by those authors. For example, in chain 1, the values for the parameters μ, σ and λ are -1.613, 0.518 and 0.669 respectively. These values are between -1.96 and 1.96. Thus, the chains produced by the MCMC algorithm based on the model studied have successfully converged into a stationary distribution.

Table 3.9 Summary of the posterior distribution based on the skew-normal degradation parameter with GaAs Laser data

Parameters	**Mean**	**SD**	**Quantiles**					$\hat{R}$
			2.5%	**25%**	**50%**	**75%**	**97.5%**	
μ	1.713	0.195	1.294	1.580	1.740	1.874	1.987	1.001
σ	0.937	0.294	0.479	0.724	0.894	1.104	1.644	1.001
λ	4.018	2.707	0.773	2.058	3.344	5.283	10.930	1.001

Finally, summary of the posterior distribution for the parameters with convergence statistics $\hat{R}$ based on the GaAs Laser data in terms of mean and standard deviations are provided in Table 3.9.

Gelman and Hill (2007) suggest that if the convergence statistics $\hat{R}$ for all the parameters are found less than 1.1, the resultant posterior distributions attained convergence. Since the values of $\hat{R}$ for all the parameters according to the results in Table 3.9 are found equal 1.001, which are less than 1.1, so the posterior distributions have attained convergence.

3.5.3 Application of Laser Degradation Data for the Bayesian Approach of Skew-Normal Linear Degradation Model based on Different Priors

In this section, the GaAs laser degradation data is applied and comparison of the performance of the informative, weakly informative and non-informative priors are made. The total number of the MCMC iterations involving the two chains is set to be 100000 where the first 50000 iterations are treated as burn-in. Again, the values of hyperparameters are assumed to follow the uniform distribution U (0, 2), following Kundu (2008). While for the gamma prior distribution, the hyperparameters are set to get the variance of less than 1, indicating the informative prior. For weakly informative prior, the scale and shape parameters are equal 0.1 and 100 respectively; thus, resulting in a large variance. In this case, the variance is 1000, which reflects the present of little information available with the prior distribution. The comparison is made based in terms of point estimate and standard deviation of the percentiles of the TTF distribution. The results found are provided in Table 3.10.

Table 3.10 PE, B and SD for certain percentiles of the skew-normal linear degradation model under informative, weakly informative and non-informative priors based on the GaAs laser degradation data

Parameters	**Informative prior**			**Weakly informative prior**			**Non-informative prior**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$t_{0.05}$ = 7.128	6.271	0.857	0.957	6.362	0.766	0.861	6.207	0.921	1.019
$t_{0.2}$ = 8.125	7.996	0.129	0.654	7.690	0.435	0.630	7.939	0.186	0.728
$t_{0.5}$ = 10.518	9.840	0.678	0.492	9.613	0.905	0.632	9.856	0.662	0.589
$t_{0.75}$ = 11.778	11.354	0.424	0.592	11.542	0.236	0.902	11.500	0.279	0.710

$t_{0.9}$ = 12.212	12.745	-0.533	0.830	13.481	-1.269	1.352	13.071	-0.859	0.970

Based on the Table 3.10, the following results are found:

i) The SD values for all the large percentiles based on informative prior are smaller than those found based on the other type of priors. For example, for 5th percentile, the values of SD are 0.492, 0.632 and 0.589 under informative, weakly informative and non-informative priors respectively.

ii) In 2 out of 5 cases, the B value under informative and weakly informative are less than those found under non-informative prior. For example, for 20th and 90th percentiles under informative prior, the B values are 0,129 and -0.533 respectively while for those found under weakly informative and non-informative priors, the B values are 0.435, -1.269 and 0.186 and -0.859 respectively.

According to above results under the application of laser degradation data, the performance of the skew-normal linear degradation model based on informative prior is more likely to be better than the performance of the skew-normal linear degradation model based on non-informative and weakly informative priors in estimated the percentiles of the TTF distribution.

3.5.4 Application of Laser Degradation Data in Comparing the Bayesian Approach of Skew-Normal and Log-logistic Linear Degradation Models

In this section, the GaAs data is applied and comparison of the Bayesian results for the TTFL under skew-normal degradation parameter and log-logistic degradation parameter based on informative prior are carried out without using the estimated value μ of 1.713 as given in Table 3.9, but with the rounding value of μ equal to 2. In order to compare the performance of those models, the point estimate, bias and standard deviation of the certain percentiles of the TTF distribution found based on skew-normal and log-logistic distributions that are indicated in Equations (3.5) and (3.11) respectively are computed. Additionally, deviance information criterion (DIC) is measured to select the best model fitted to the real data. Spiegelhalter et al. (2002) define the DIC for the vector of parameters interest, denoted as $\boldsymbol{\theta}$, as the following:

$$DIC = \bar{D}(\boldsymbol{\theta}) + P_D \qquad \ldots(3.22)$$

where D is the deviance is defined by – 2log (l), l is the likelihood function, $\bar{D}$ is the posterior deviance mean which is given based on the values of the estimated parameters of the posterior distribution, and P_D is the effective number of parameters of the model which is defined as $\bar{D}(\boldsymbol{\theta}) - D(\bar{\boldsymbol{\theta}})$ where $D(\bar{\boldsymbol{\theta}})$ is the deviance which is found by finding the mean of each value of the estimated parameters in the posterior distribution. Results of the comparison are provided in Table 3.11.

Table 3.11 PE, B and SD of the percentiles and DIC for the skew-normal and log-logistic linear degradation models for GaAs laser degradation data

Percentiles	**Skew-normal linear degradation model**			**Log-logistic linear degradation model**		
	PE	**B**	**SD**	**PE**	**B**	**SD**
$t_{0.05}$= 7.128	6.696	0.433	0.734	6.055	1.074	0.713
$t_{0.2}$= 8.125	8.379	-0.254	0.421	7.693	0.432	0.537
$t_{0.5}$= 10.518	10.144	0.373	0.249	9.545	0.973	0.370
$t_{0.75}$= 11.778	11.560	0.218	0.405	11.345	0.433	0.556
$t_{0.9}$= 12.212	12.835	-0.623	0.636	13.507	-1.295	1.124
DIC		66.483			68.286	

Comparison of the skew-normal and the log-logistic linear degradation models in Table 3.11 indicates the following results:

i) In 4 out of 5 cases, the SD for the estimated percentiles of skew-normal linear degradation model is smaller than the SD for the estimated percentiles of log-logistic linear degradation model. For example, for 75th percentiles, the SD values are 0.405 and 0.556 under skew-normal and log-logistic linear degradation models respectively.

ii) All the B values of the skew-normal linear degradation model are smaller than those found in the log-logistic linear degradation model. For example, for 75th percentiles, the B values are 0,218 and 0.433 under skew-normal and log-logistic linear degradation models respectively.

iii) The PE of the percentiles of skew-normal linear degradation model is found closer to the observed value of the percentiles reported than those values found for the log-logistic linear degradation model. This finding can be proven by using the linear interpolation of the observed values. For example, consider the interpolated value of the 75th percentile of the TTF values in Table 3.10, i.e. 11.778. Corresponding to the estimated values in the Table 3.11, we have $\hat{t}_{0.75} = 11.560$ for the skew-normal linear degradation model while for the log-logistic linear degradation model $\hat{t}_{0.75} = 11.345$, and the SD values are 0.405 and 0.556 respectively.

iv) The deviance information criterion for skew-normal linear degradation model is slightly smaller than the deviance information criterion for log-logistic linear degradation model.

The above results indicate that the estimated value of the parameter for the skew-normal linear degradation model is more precise and hence it is more appropriate to use them as compared to that found based on log-logistic linear degradation model.

3.6 CONCLUSION

This Chapter presents the Bayesian approach to estimate the parameters and the percentiles of the TTFL with the degradation parameter follows either skew-normal or log-logistic distributions based on the informative, weakly informative and non-informative priors. From the simulation results, we conclude that the Bayesian approach which considers informative prior outperformed the Bayesian model found under weakly informative and non-informative prior distributions, especially in the case of small sample size, while for the large sample size both results are found to be close. Additionally, comparison of the skew-normal and log-logistic linear degradation models is presented in terms of point estimate, bias, standard deviation and deviance information criterion. It is found that the linear degradation model with the degradation parameter following the skew-normal distribution presents as a better alternative model than the log-logistic linear degradation model for describing the degradation data. These results found more apparent in the simulated data and real data.

CHAPTER IV

BAYESIAN APPROACH FOR TTF DISTRIBUTION BASED ON POWER DEGRADATION MODEL

4.1 INTRODUCTION

In this chapter, a power degradation model is used to estimate the TTF distribution. This model is flexible and can accommodate various degradation behaviours; thus, it is widely used in many fields such as material science, reliability engineering and electronics for predicting the RUL of components. The degradation parameter of the power degradation model is assumed to follow the skew-normal distribution as in the first case while in the second case the degradation parameter is assumed to follow the log-logistic distribution. The Bayesian approach is applied to estimate the parameters and percentiles of the TTF distribution. As in Chapter 3, the simulation study is implemented by MCMC based on R programming with JAGS platform. The models considered are applied on the simulated data and the NASA turbofan Jet engine dataset. In particular, for skew-normal power degradation model, the analysis of convergence is provided.

4.2 TTF DISTRIBUTION FOR POWER DEGRADATION MODEL (TTFP)

The actual path in Equation (2.2) can be written in several forms of mathematical expression. In this study, a nonlinear degradation model which consists of a power function is expressed as the following:

$$D_f = \varphi T^{\frac{1}{\beta}} \quad \ldots(4.1)$$

In Equation (4.1), T is defined as a function of β as the following:

$$T = \left(\frac{D_f}{\varphi}\right)^{\beta} \qquad \text{...(4.2)}$$

The cdf of the TTFP, denoted as F_{T-P}, can be derived as follows:

$$\begin{aligned} F_{T-P}(t) &= \Pr(T \leq t) \qquad \text{...(4.3)} \\ &= \Pr\left(\left(\frac{D_f}{\varphi}\right)^{\beta} \leq t\right) \\ &= \Pr\left(\beta \, \ln\left(\frac{D_f}{\varphi}\right) \leq \ln t\right) \end{aligned}$$

Then,

$$F_{T-P}(t) = G_{\beta}\left(\frac{\ln t}{\ln\left(\frac{D_f}{\varphi}\right)}\right), \ \frac{D_f}{\varphi} > 1 \text{ and } t > 1 \qquad \text{...(4.4)}$$

By taking the derivative of both sides of Equation (4.4) with respect to *t*, the pdf of the TTFP, denoted as f_{T-P}, is obtained as the following:

$$\frac{d\,(F_{T-P}(t))}{dt} = \frac{d\left(G_{\beta}\left(\frac{\ln t}{\ln\left(\frac{D_f}{\varphi}\right)}\right)\right)}{dt} \qquad \text{...(4.5)}$$

$$f_{T-P}(t) = \left(\frac{1}{t\ln\left(\frac{D_f}{\varphi}\right)}\right) g_{\beta}\left(\frac{\ln t}{\ln\left(\frac{D_f}{\varphi}\right)}\right) \qquad \text{...(4.6)}$$

As seen in Equations (4.4) and (4.6), the cdf and the pdf of the TTF distribution are dependent on the distribution of degradation parameter β. Based on the assumptions of the distribution of the degradation parameter β as in Chapter 3, the derivation of TTFP is given in the Sections 4.2.1 and 4.2.2.

4.2.1 Power Degradation Model with Skew-Normal Degradation Parameter

As Chapter 3, β is assumed to follow the skew-normal distribution with parameters μ, σ and λ. By modifying Equations (4.4) and (4.6) based on the Equations (2.9) and

(2.10) respectively, the cdf and the pdf of the TTFP with skew-normal degradation parameter, denoted respectively as F_{T-PSN} and f_{T-PSN}, can be given as:

$$F_{T-PSN}(t) = \Phi\left(\frac{\ln(t)-\mu\ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ln\left(\frac{D_f}{\varphi}\right)}\right) - 2T\left(\frac{\ln(t)-\mu\ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ln\left(\frac{D_f}{\varphi}\right)}, \lambda\right) \qquad \ldots(4.7)$$

and

$$f_{T-PSN}(t) = \frac{2}{\sigma\, t\ln\left(\frac{D_f}{\varphi}\right)}\,\phi\left(\frac{\ln(t)-\mu\ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ln\left(\frac{D_f}{\varphi}\right)}\right)\Phi\left(\lambda\left(\frac{\ln(t)-\mu\ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \qquad \ldots(4.8)$$

The 100 r^{th} percentile of the TTFP with skew-normal degradation parameter, denoted as t_{r-PSN}, is determined by solving Equation (4.4) for *t*.

Let

$$G_\beta\left(\frac{\ln t}{\ln\left(\frac{D_f}{\varphi}\right)}\right) = r \qquad \ldots(4.9)$$

By taking the inverse function for both sides, we get

$$\frac{\ln t}{\ln\left(\frac{D_f}{\varphi}\right)} = G_\beta^{\ -1}(r) \qquad \ldots(4.10)$$

Then,

$$t_{r-PSN} = e^{ln\left(\frac{D_f}{\varphi}\right) G_\beta^{-1}(r)} \qquad \ldots(4.11)$$

Equation (4.11) is solved numerically to find the value of t_{r-PSN}.

Suppose a random sample of size n from the TTF distribution with parameters μ, σ, λ and φ in Equation (4.8) be denoted as $t_1, t_2, \dots, t_n$. Then, the likelihood function for the random sample can be written as the following:

$$L(\mu, \sigma, \lambda, \varphi; \boldsymbol{t}) =$$
$$\left(\frac{2}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)^n \prod_{i=1}^{n} \frac{1}{t_i} \phi\left(\frac{\ln t_i - \mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right) \Phi\left(\lambda\left(\frac{\ln t_i - \mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \quad \dots(4.12)$$

The likelihood function as given in Equation (4.12) is considered in order to determine the posterior distribution of the parameters of the TTFP with the degradation parameter assumed to follow skew-normal distribution.

4.2.2 Power Degradation Model with Log-logistic Degradation Parameter

In this section, the TTF distribution is derived based on the power degradation model in Equation (4.1) by assuming that the degradation parameter β follows the log-logistic distribution. The cdf and the pdf of the TTF distribution, denoted as F_{T-PL} and f_{T-PL} respectively, are given by modifying Equations (4.4) and (4.6) using Equations (2.12) and (2.13) respectively as the following:

$$F_{T-PL}(t) = \frac{1}{1+\left(\frac{\ln t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}} \quad \dots(4.13)$$

and

$$f_{T-PL}(t) = \frac{\left(\frac{\omega}{\alpha\, t \ln\left(\frac{D_f}{\varphi}\right)}\right)\left(\frac{\ln t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega-1}}{\left(1+\left(\frac{\ln t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega}\right)^2} \quad \dots(4.14)$$

The 100 r^{th} percentile of the TTFP with log-logistic degradation parameter, denoted as t_{r-PL}, can be determined by solving Equation (4.13) for t_{r-PL}. Let

$$\frac{1}{1+\left(\frac{\ln t_{r-PL}}{\alpha\ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}} = r \qquad \ldots(4.15)$$

Then,

$$\left(\frac{\alpha\ln\left(\frac{D_f}{\varphi}\right)}{\ln t_{r-PL}}\right)^{\omega} = \frac{1-r}{r} \qquad \ldots(4.16)$$

$$(\ln t_{r-PL})^{\omega} = \left(\alpha\ln\left(\frac{D_f}{\varphi}\right)\right)^{\omega}\frac{r}{1-r} \qquad \ldots(4.17)$$

By taking the root of ω for both sides, we get

$$\ln t_{r-PL} = \alpha\ln\left(\frac{D_f}{\varphi}\right)\left(\frac{1-r}{r}\right)^{\frac{-1}{\omega}} \qquad \ldots(4.18)$$

Then,

$$t_{r-PL} = e^{\alpha\ln\left(\frac{D_f}{\varphi}\right)\left(\frac{1-r}{r}\right)^{\frac{-1}{\omega}}} \qquad \ldots(4.19)$$

Let $t_1, t_2, \ldots, t_n$ be a random sample of size n having the pdf given by Equation (4.14). The likelihood function of the parameters α, ω and φ is given as follows:

$$L(\alpha,\omega,\varphi\,;\boldsymbol{t}) = \frac{\omega^n}{\left(\alpha\ln\left(\frac{D_f}{\varphi}\right)\right)^{2n+\omega-1}}\prod_{i=1}^{n}\frac{(\ln t_i)^{\omega-1}}{t_i\left(1+\left(\frac{\ln t_i}{\alpha\ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega}\right)^2} \qquad \ldots(4.20)$$

In the Bayesian approach, the likelihood function in Equation (4.20) is applied to obtain the posterior distribution of the parameters α, ω and φ.

4.3 BAYESIAN MODELING OF TTFP

The Bayesian approach is applied to estimate the parameters of the TTFP and its percentiles. The sensitivity of the choices of the different prior distributions for estimating the parameters and the percentiles of the TTF distribution is studied when the degradation parameter is assumed to follow the skew-normal by considering informative and non-informative prior distributions. According to Equation (2.17), the joint posterior density function of the TTFP is presented for different assumptions of the degradation parameter which are skew-normal distribution or log-logistic distribution.

4.3.1 Posterior Density of TTFP with Skew-Normal Degradation Parameter

Based on the likelihood function in Equation (4.12) and by applying the same assumptions of the prior distribution for the parameters μ, σ, λ and φ as provided in Chapter 3, the joint posterior density function of TTFP with skew-normal degradation parameter, denoted as $\pi_{PSN}(\mu, \sigma, \lambda, \varphi|\boldsymbol{t})$, is found to be proportional to:

$$\frac{(\lambda)^{a-1}\, e^{\frac{-\lambda}{b}}}{c\; d\; k\; b^{a}\, \Gamma(a)}\left(\frac{2}{\sigma\; ln\left(\frac{D_f}{\varphi}\right)}\right)^{n}$$
$$\prod_{i=1}^{n}\frac{1}{t_i}\phi\left(\frac{\ln t_i-\mu\; ln\left(\frac{D_f}{\varphi}\right)}{\sigma\; ln\left(\frac{D_f}{\varphi}\right)}\right)\Phi\left(\lambda\left(\frac{\ln t_i-\mu\; ln\left(\frac{D_f}{\varphi}\right)}{\sigma\; ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \qquad \ldots(4.21)$$

Again, as in Chapter 3, the marginal posterior densities are required in order to determine the parameter estimates. The marginal posterior density functions $\pi(\mu|\boldsymbol{t})$, $\pi(\sigma|\boldsymbol{t})$, $\pi(\lambda|\boldsymbol{t})$ and $\pi(\varphi|\boldsymbol{t})$ involve a complex integration. Thus, the parameters are estimated by implementing the MCMC method available in JAGS software under R programming.

4.3.2 Posterior Density of TTFP with Log-logistic Degradation Parameter

The prior distributions for the parameters of the TTFP with the log-logistic degradation parameter are assumed in such a way that the log-logistic power degradation model is comparable with skew-normal power degradation model. Accordingly, we have $\alpha \sim$ uniform (0, *c*), $\omega \sim$ gamma (*a, b*) and $\varphi \sim$ uniform (0, *k*). Then, the joint posterior density under TTFP with log-logistic degradation parameter, denoted as $\pi_{PL}(\alpha, \omega, \varphi|\boldsymbol{t})$, is proportional to:

$$\frac{\omega^{n+u-1}\, e^{\frac{-\omega}{b}}}{c\, k\, b^{a}\, \Gamma(a)\left(\alpha \ln\left(\frac{D_f}{\varphi}\right)\right)^{2n+\omega-1}} \prod_{i=1}^{n} \frac{(\ln t_i)^{\omega-1}}{t_i\left(1+\left(\frac{\ln t_i}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega}\right)^{2}} \qquad \ldots(4.22)$$

The marginal posterior densities for the parameters α, ω and φ based on the joint posterior distribution of the TTF distribution given in Equation (4.22) are also quite complicated. Thus, estimation of the parameters α, ω and φ is carried out based on the MCMC method available in the JAGS platform written in R.

4.4 SIMULATION STUDY OF TTFP

The Bayesian approach is carried out based on the MCMC method under JAGS platform to estimate the parameters of the distributions in Equations (4.7) and (4.13) and their percentiles. The data of the TTFP are simulated randomly based on Equations (4.11) and Equation (4.19) for the sample size *n* = 30, 60 and 200. Additionally, for shape parameters, the prior distribution of the parameters is assumed gamma distribution while for the other parameters, uniform distribution is considered. The MCMC is applied involving 100000 iterations where half of them is used as burn-in.

4.4.1 Bayesian Analysis for TTFP with Skew-Normal Degradation Parameter Based on Several Different Priors

The simulation study is implemented by using the same true parameter and initial values as in Chapter 3. The informative, weakly informative and non-informative are considered particularly for TTFP with skew-normal degradation parameter. Under the choices of informative and weakly informative priors, the informative gamma prior is assumed for the shape parameter λ, i.e. $\lambda \sim \Gamma(2, 2)$, while in the case of weakly informative gamma prior, $\lambda \sim \Gamma(0.1, 100)$. In addition, in the case of non-informative priors, the distributions for all the parameters are assumed uniform. The simulation results are provided in Tables 4.1 to 4.3.

From the results found in the Tables 4.1 to 4.3, the following points are noted:

i) For sample of size n = 30, in 4 out of 9 cases, the SD values of estimated parameters and percentiles determined based on informative prior are smaller than those found based on the other priors. For example, for the scale parameter σ, the values of SD are 0.857, 1.007 and 0.895 under informative, weakly informative and non-informative priors respectively.

ii) For the sample of sizes n = 60 and 200, for all types of prior all the SD values are found to be quite close. For example, when the sample size n = 60, for 5^{th} percentile, the values of SD are 0.418, 0.424 and 0.420 under informative, weakly informative and non-informative priors respectively.

iii) The SD values of the estimated parameters and the percentiles for different choices of the prior of the skew-normal power degradation model decrease as n increase for all sample sizes. For example, when the sample size n = 30, 60 and 200, for 5^{th} percentile, the SD values are 0.599, 0.418 and 0.316 under informative prior respectively.

iv) For most cases of the sample sizes, the PE values of the estimated parameters and the percentiles of the skew-normal power degradation model under different types of priors are found to be quite close to each other. For example, when the sample size n = 200, for 5^{th} percentile, the PE values are 2.423, 2.399 and 2.434 under informative, weakly informative and non-informative priors respectively.

Table 4.1 PE, B and SD for the parameter and the percentiles of the TTFP with skew-normal degradation parameter based on informative, weakly informative and non-informative gamma prior for n = 30

Parameters	**Informative prior**			**Weakly Informative prior**			**Non-informative prior**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	1.176	-0.176	0.476	1.326	-0.326	0.478	1.162	-0.162	0.475
$\sigma = 2$	2.473	-0.473	0.857	1.740	0.260	1.007	2.436	-0.436	0.895
$\lambda = 3$	2.847	0.153	1.600	1.892	1.108	3.523	2.786	0.214	1.361
$\varphi = 6$	7.261	-1.261	2.411	5.483	0.517	2.765	7.115	-1.115	2.526
$t_{0.05} = 2.251$	1.822	0.428	0.599	1.633	0.617	0.642	1.829	0.422	0.604
$t_{0.2} = 5.395$	4.691	0.703	1.330	4.755	0.640	1.535	4.698	0.697	1.321
$t_{0.5} = 16.812$	15.884	0.928	4.968	16.801	0.011	5.292	15.900	0.912	4.976
$t_{0.75} = 53.192$	53.247	-0.056	23.666	52.563	0.629	21.010	53.329	-0.138	23.694
$t_{0.9} = 174.983$	189.669	-14.685	141.989	165.639	9.344	115.060	189.522	-14.539	140.810

Table 4.2 PE, B and SD for the parameter and the percentiles of the TTFP with skew-normal degradation parameter based on informative, weakly informative and non-informative gamma prior for n = 60

Parameters	**Informative prior**			**Weakly Informative prior**			**Non-informative prior**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	0.976	0.024	0.397	0.944	0.056	0.406	1.023	-0.023	0.387
$\sigma = 2$	2.740	-0.740	0.824	2.730	-0.730	0.834	2.792	-0.792	0.800
$\lambda = 3$	4.837	-1.837	1.721	5.326	-2.326	2.702	4.383	-1.383	1.073
$\varphi = 6$	7.368	-1.368	2.454	7.264	-1.264	2.476	7.597	-1.597	2.334
$t_{0.05} = 2.251$	2.148	0.102	0.418	2.144	0.106	0.424	2.141	0.110	0.420
$t_{0.2} = 5.395$	4.657	0.738	0.785	4.602	0.793	0.792	4.699	0.696	0.789
$t_{0.5} = 16.812$	15.045	1.767	3.074	15.011	1.801	3.107	15.054	1.758	3.047
$t_{0.75} = 53.192$	53.835	-0.644	16.328	54.561	-1.369	16.975	53.088	0.103	15.587
$t_{0.9} = 174.983$	206.119	-31.136	95.204	212.809	-37.825	102.940	199.632	-24.649	87.139

Table 4.3 PE, B and SD for the parameter and the percentiles of the TTFP with skew-normal degradation parameter based on informative, weakly informative and non-informative gamma prior for n = 200

Parameters	Informative prior			Weakly Informative prior			Non-informative prior		
	PE	B	SD	PE	B	SD	PE	B	SD
$\mu = 1$	1.374	-0.374	0.417	1.399	-0.399	0.382	1.392	-0.392	0.391
$\sigma = 2$	2.464	-0.464	0.828	2.418	-0.418	0.770	2.538	-0.538	0.774
$\lambda = 3$	2.854	0.145	0.786	2.721	0.279	0.837	2.955	0.045	0.836
$\varphi = 6$	7.164	-1.164	2.492	7.162	-1.162	2.243	7.394	-1.394	2.233
$t_{0.05} = 2.251$	2.423	-0.173	0.316	2.399	-0.148	0.326	2.434	-0.183	0.313
$t_{0.2} = 5.395$	6.044	-0.649	0.659	6.075	-0.680	0.669	6.016	-0.621	0.662
$t_{0.5} = 16.812$	19.477	-2.665	2.352	19.678	-2.866	2.417	19.367	-2.555	2.366
$t_{0.75} = 53.192$	62.263	-9.071	9.062	62.379	-9.188	8.969	62.321	-9.129	9.081
$t_{0.9} = 174.983$	206.472	-31.488	43.728	204.214	-29.231	42.757	208.414	-33.431	44.207

According to the above points, under small sample size, the Bayesian approach based on the informative prior is more precise for estimating the parameters and the percentiles of TTFP while for the large sample size, the Bayesian approach based on the different types of priors are found to be quite close.

4.4.2 Comparison Between Log-logistic and Skew-Normal Power Degradation Models

A comparison between the posterior densities given in Equations (4.21) and (4.22) is made in terms of the PE, SD and DIC. In the case of degradation parameter following the skew-normal model, the location parameter μ is not estimated. The true values of the parameters and hyperparameters in Section 4.4.1 are used. The results are provided in Tables 4.4 and 4.5.

From the results found in Tables 4.4 and 4.5, the following points are noted:

i) The SD values of the estimated parameters and percentiles in both tables decrease when sample size increase. For example, for the shape parameter λ, the SD values are 1.809, 1.421 and 0.771 for sample sizes n = 30, 60 and 200 respectively.

ii) The SD values of the percentiles in both tables increase when r increase. For example, under TTFP with skew-normal degradation parameter and sample size n = 200, for 5^{th} up to 90^{th} percentiles, the SD values are 0.256, 0.502, 1.635, 6.932 and 34.016 respectively.

iii) For sample of size n = 30, in 5 out of 8 cases, the SD and B values of the estimated parameters and percentiles under skew-normal power degradation model are smaller than those found for log-logistic power degradation model. For example, under skew-normal power degradation model, for the scale parameter and 5^{th} percentiles, the SD values are 0.575 and 0.498 respectively while for those found under log-logistic power degradation model, the SD values are 0.896 and 0.538 respectively.

iv) In 6 out of 9 cases, for the different sample sizes, SD values of the parameters of the TTFP with skew-normal degradation parameter are found to be smaller than those found based on the TTFP with log-logistic degradation parameter. For example, for the sample size $n = 30$, the SD values of the parameters σ and φ of the TTFP with skew-normal degradation parameter are 0.575 and 1.297 respectively, while for the parameters α and φ of the TTFP with log-logistic degradation parameter, the SD values are 0.896 and 2.763 respectively.

v) For sample of sizes $n = 60$ and 200, 8 out of 10 cases of the SD values of the percentiles of the TTFP with log-logistic degradation parameter are found to be smaller than those found based on the TTFP with skew-normal degradation parameter. For example, for the sample size $n = 60$, the SD values for the estimated percentiles of the TTFP with log-logistic degradation parameter are 0.276, 0.587, 1.909, 10.014 and 135.340, while for the TTFP with skew-normal degradation parameter, the SD values are 0.491, 0.978, 3.924, 20.642 and 118.882 respectively.

vi) In 5 out of 9 cases of different sample sizes, the PE of the percentiles of TTFP with log-logistic degradation parameter are found to be closer to the true value than those found under the TTFP with skew-normal degradation.

According to the above notes, under small sample size, it is clear that the Bayesian approach for the power degradation model with the skew-normal degradation parameter outperformed the power degradation model with the log-logistic degradation parameter. While for large sample size and small r^{th} percentile, the performance of Bayesian approach for the power degradation model with log-logistic degradation parameter is slightly better. However, for large sample size and large r^{th} percentile the performance of power degradation model with skew-normal degradation parameter is better.

Table 4.4 True parameter values, PE, B and SD for the parameters and percentiles of the TTFP based on the skew-normal degradation parameter

Parameters	***n* = 30**			***n* = 60**			***n* = 200**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\sigma = 2$	3.169	-1.169	0.575	2.500	-0.500	0.656	2.168	-0.168	0.505
$\lambda = 3$	4.581	-1.581	1.809	4.117	-1.117	1.421	3.005	-0.005	0.771
$\varphi = 6$	8.026	-2.026	1.297	6.714	-0.714	1.449	6.117	-0.117	1.185
$t_{0.05} = 2.251$	2.035	0.216	0.498	2.380	-0.129	0.491	2.430	-0.180	0.256
$t_{0.2} = 5.395$	4.854	0.540	0.999	5.524	-0.129	0.978	5.148	0.247	0.502
$t_{0.5} = 16.812$	17.982	-1.170	5.677	18.543	-1.731	3.924	15.175	1.638	1.635
$t_{0.75} = 53.192$	74.242	-21.050	37.095	67.336	-14.145	20.642	48.172	5.019	6.932
$t_{0.9} = 174.983$	332.011	-157.028	250.066	260.903	-85.920	118.882	160.661	14.323	34.016

Table 4.5 True parameter values, PE, B and SD for the parameters and percentiles of the TTFP based on the log-logistic degradation parameter

Parameters	***n* = 30**			***n* = 60**			***n* = 200**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\alpha = 2$	2.466	-0.466	0.896	2.386	-0.386	0.905	2.591	-0.591	0.854
$\omega = 3$	3.086	-0.086	0.457	3.003	-0.003	0.325	2.998	0.002	0.177
$\varphi = 6$	6.345	-0.345	2.763	7.019	-1.019	2.931	7.335	-1.335	2.598
$t_{0.05} = 2.465$	2.849	-0.384	0.538	2.411	0.054	0.276	2.517	-0.051	0.164
$t_{0.2} = 4.558$	5.669	-1.110	1.270	4.388	0.170	0.587	4.720	-0.162	0.359
$t_{0.5} = 11.111$	15.544	-4.433	4.858	10.566	0.545	1.909	11.793	-0.682	1.195
$t_{0.75} = 32.229$	55.447	-23.218	35.867	31.124	1.104	10.014	35.559	-3.331	6.026
$t_{0.9}$=149.714	539.986	-390.271	4189.942	164.397	-14.683	135.340	179.670	-29.955	60.427

4.5 NASA TURBOFAN JET ENGINE DATASET (NTJE)

4.5.1 Describing the NTJE Data

The data considered in this study is run-to-failure simulated data of turbofan jet engines which are available from NASA. The experiment of this NASA study is implemented for four different types of engines, denoted as FD001, FD002, FD003 and FD004, where each engine set contains a fleet of engines under different manufacturing operations, degrees of initial wear and run operational settings. In the assessment of the faults, 21 sensor measurements are made for each engine. The sensor measurements consist of two sub-data sets which are training and test data. The training data are obtained by letting all engines follow process of the operation until failure while the test data are found until all engines arrive at a certain point of time before the failure. Accordingly, the NTJE data consists of 26 columns of numbers which are organized as engine number, cycles, operation setting 1,2 and 3 and 21 columns for sensor measurements. In addition, the vector of RUL for the 100 engines are provided for used in this study. More details on the data can be found in (Saxena et al. 2008 & Chen et al. 2020).

4.5.2 Convergence Analysis for NTJE Data

The total number of the MCMC iterations involving the two chains are set to be 100000 where the first 50000 iterations are treated as burn-in. Following (Kundu 2008), the values of the hyperparameters are all assumed equal to 2. To check for the convergence of the series produced based on the MCMC simulation, graphs of the trace plots, posterior density functions and autocorrelation of the parameters μ, σ, λ and φ are determined particularly for TTFP with skew-normal degradation parameter and presented in Figure 4.1.

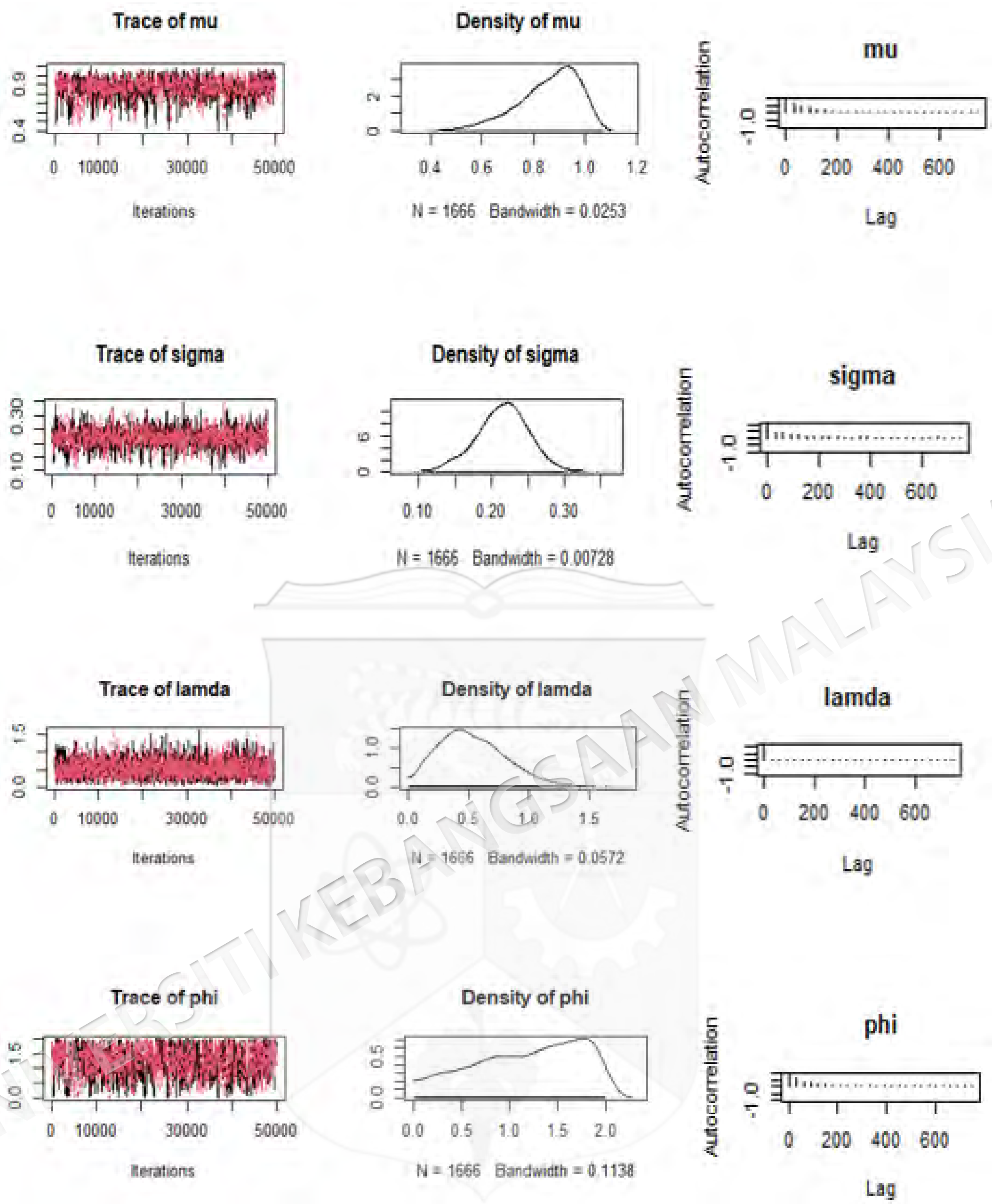

Figure 4.1 Trace plot, posterior density function and autocorrelation of the parameters μ, σ, λ and φ based on the Bayesian analysis of NTJE data under skew-normal power degradation model

As shown in Figure 4.1, the proposed model converges to the desired distribution satisfactorily well. Also, it appears that the autocorrelation graph with thinning of 30 reaches zero rather quickly, indicating that the chain is mixing adequately. Additionally, the values of the potential scale reduction factor of Gelman-Rubin and Geweke's stationarity test for convergence assessment which are provided in Table 4.6 further support the case of convergence as mentioned in Chapter 3.

Table 4.6 Geweke's stationarity test and the potential scale reduction factor to assess convergence of the MCMC chains for the parameters μ, σ, λ and φ of the posterior distribution based on NTJE data under skew-normal power degradation model

Parameters	**Geweke's stationarity test**			**Psrf**
	Chain 1	**Chain 2**	**Chain 3**	
μ	-0.26	1.24	-1.22	1.01
σ	-0.68	0.94	-1.29	1.00
λ	-0.85	-0.98	0.15	1.01
φ	-0.16	0.63	-1.09	1.01

As mentioned in Chapter 3, if the values of the Geweke's stationarity test are all between -1.96 and 1.96, the chain is considered to converge to a stationary distribution. Since the values of psrf for all the parameters are less than 1.1, the chain is said to converge to a stationary distribution. It is apparent that the results from Table 4.6 satisfy the desired properties of convergence. Thus, the chains produced by the MCMC algorithm based on the model studied have successfully converged into a stationary distribution.

The summary of the posterior distribution for the parameters with $\hat{R}$ based on the NTJE data in terms of mean and SD are provided in Table 4.7.

Table 4.7 Summary of the posterior distribution based on the TTFP with skew-normal degradation parameter under NTJE data

Parameters	**Mean**	**SD**	**Quantiles**					$\hat{R}$
			2.5%	**25%**	**50%**	**75%**	**97.5%**	
μ	0.855	0.119	0.580	0.783	0.876	0.944	1.033	1.006
σ	0.215	0.036	0.141	0.192	0.216	0.238	0.287	1.003
λ	0.537	0.276	0.100	0.331	0.507	0.708	1.149	1.002
φ	1.181	0.537	0.151	0.754	1.240	1.647	1.966	1.007

The convergence statistics $\hat{R}$ for all the parameters are found to be less than 1.1, which indicated that the posterior distribution obtained convergence (Gelman and Hill 2007).

4.5.3 Application of NTJE Data for the Bayesian Approach of Skew-Normal Power Degradation Model based on Different Priors

Here, the analysis of NTJE data involves the sensitivity analysis of the different choices of the priors and the two choices of the distributions of degradation parameter. For the first case, the comparison is made to determine which type of prior contributes to a more precise results in terms of PE and SD. All hyperparameters have the same value equal to 2 for informative and non-informative priors while in the weakly informative the value of the hyperparameters of the gamma prior is the same as in the case of simulated data. The results found are provided in Table 4.8.

Table 4.8 PE and SD of the percentiles of the TTFP with skew-normal degradation parameter based on the NTJE data

Parameters	Informative prior		Weakly informative prior		Non- informative prior	
	PE	SD	PE	SD	PE	SD
$t_{0.05} = 9.95$	14.454	1.865	14.471	1.875	14.514	1.887
$t_{0.2} = 27.60$	28.430	2.815	28.445	2.838	28.580	2.829
$t_{0.5} = 86.00$	58.328	5.013	58.366	5.085	58.507	5.015
$t_{0.75} = 112.25$	104.680	10.112	104.842	10.318	104.475	10.124
$t_{0.9} = 126.20$	178.615	22.326	179.173	22.343	176.920	21.526

As noted from Table 4.8, the results found based on the different priors are quite close. Based on the large sample size of 100 for the NTJE data, there are no big difference in the values of the estimated parameters and the percentiles of the TTF distribution found. The SD of the estimated parameters and the percentiles based on informative prior is slightly smaller than those found based on other priors.

4.5.4 Application of NTJE Data in Comparing the Bayesian Approach of Skew-Normal and Log-logistic Power Degradation Models

The results for the comparison between the adequacy of the models for describing NTJE data are presented in terms of PE, SE and DIC, where all hyperparameters of the prior distributions are all assumed equal to 2, are provided in Table 4.9.

Table 4.9 PE and SE of the percentiles of the TTFP with skew-normal and log-logistic degradation parameter and DIC for the models using the NTJE data

Percentiles	Skew-normal power degradation model		Log-logistic power degradation model	
	PE	SE	PE	SE
$t_{0.05} = 9.95$	14.450	0.187	16.620	0.201
$t_{0.2} = 27.60$	28.430	0.282	31.410	0.320
$t_{0.5} = 86.00$	58.330	0.505	62.610	0.593
$t_{0.75} = 112.25$	104.700	1.031	119.600	1.421
$t_{0.9} = 126.20$	178.600	2.233	257.600	4.626
DIC	1066.900		1085.300	

As noted in Table 4.9, the performance of the skew-normal power degradation model is more precise based on the analysis of NJTE data because the results indicate smaller SE for the percentile estimates found for the skew-normal power degradation model as opposed to those found for the log-logistic power degradation model. In addition, the skew-normal power degradation model has a smaller value of DIC than log-logistic power degradation model. So, it is a better model in terms of goodness of fit for modelling the NJTE data.

4.6 CONCLUSION

TTF distribution for the power degradation model is derived based on two assumptions of the distribution of the degradation parameter which are the skew-normal and the log-logistic distributions. By using the Bayesian technique, the parameters and percentiles of the TTF distribution are estimated. Based on some convergence tests, the chains are found mixing well and attaining convergence for the posterior distribution of the parameters of the skew-normal distribution and the fixed-effect parameter. The Bayesian estimator of the parameters and the percentiles of TTFP based on the skew-normal and the log-logistic degradation parameter are compared using the simulated data and NTJE data. Based on point estimate, standard deviation, standard error, and deviance information criteria, the Bayesian estimation for the skew-normal power degradation model outperformed the Bayesian estimation for the other model because the parameters of the TTF distribution and its percentiles are estimated with higher precision.

CHAPTER V

BAYESIAN APPROACH FOR TTF DISTRIBUTION BASED ON EXPONENTIAL DEGRADATION MODEL

5.1 INTRODUCTION

In addition to linear degradation with respect to time, it is also possible that the degradation over time can take on an exponential form. While there are various mathematical functions that can represent the non-linear general path degradation model, in this chapter, the exponential general degradation model which assumes that the degradation process follows an exponential function over time, i.e. the degradation level increases exponentially with time, that have been reported by Siju and Kumar (2018) is considered here. This model is commonly used to describe degradation processes in reliability engineering, such as wear, corrosion and fatigue (Si et al. 2012). The Bayesian approach is applied on the exponential degradation path model for estimating the parameters and the percentiles of the TTF distribution where the degradation parameter is assumed to follow the skew-normal distribution. Sensitivity analysis involving several choices of the prior distributions is considered in the Bayesian analysis. The results obtained are compared by using MCMC simulation under the JAGS platform in terms of bias and standard deviation. Additionally, under the Bayesian approach, a simulation study is carried out to compare the performance of the skew-normal exponential degradation model with the log-logistic exponential degradation model. For illustration using real data, the models are applied to the fatigue-crack data and the results found are compared in terms of PE, SD and DIC.

5.2 TTF DISTRIBUTION FOR EXPONENTIAL DEGRADATION MODEL (TTFE)

The actual path of general degradation model given in Equation (2.2) can be expressed as exponential function given as:

$$D_f = \varphi e^{\frac{T}{\beta}} \qquad \ldots(5.1)$$

By solving for T as a function of β, Equation (5.1) can be given as

$$T = \beta \ ln\left(\frac{D_f}{\varphi}\right) \qquad \ldots(5.2)$$

Based on the definition of the cdf, the cdf of the TTFE based on Equation (5.2), denoted as F_{T-E}, is given by

$$F_{T-E}(t) \ = G_\beta\left(\frac{t}{ln\left(\frac{D_f}{\varphi}\right)}\right) \qquad \ldots(5.3)$$

By taking the derivative of Equation (5.3) with respect to *t,* the pdf of the TTFE, denoted as f_{T-E}, is given by

$$f_{T-E}(t) = \frac{1}{ln\left(\frac{D_f}{\varphi}\right)} \ g_\beta\left(\frac{t}{ln\left(\frac{D_f}{\varphi}\right)}\right) \qquad \ldots(5.4)$$

It is clear that the cdf and pdf of the TTFE in Equations (5.3) and (5.4) depend on the distribution of the degradation parameter β. It is assumed that β follows either skew-normal or log-logistic distributions.

5.2.1 Exponential Degradation Model with Skew-Normal Degradation Parameter

It can be shown by modifying Equation (5.3) based on Equation (2.9), the cdf of the TTFE with skew-normal degradation parameter, denoted as F_{T-ESN}, is given as

$$F_{T-ESN}(t) = \Phi\left(\frac{t-\mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right) - 2T\left(\frac{t-\mu\, ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}, \lambda\right) \qquad \ldots(5.5)$$

By modifying the Equation (5.4) and using the definition of skew-normal distribution, the pdf of the TTFE with skew-normal degradation parameter, denoted as f_{T-ESN}, is given as follows:

$$f_{T-ESN}(t) = \frac{2}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\ \phi\left(\frac{t-\mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)\Phi\left(\ \lambda\left(\frac{t-\mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \qquad \ldots(5.6)$$

The Equations (5.5) and (5.6) are respectively the cdf and the pdf of skew-normal distribution with location parameter $\mu\ ln\left(\frac{D_f}{\varphi}\right)$, scale parameter $\sigma\ ln\left(\frac{D_f}{\varphi}\right)$ and shape parameter λ.

To determine the 100 r^{th} percentiles of the TTFE with skew-normal degradation parameter, denoted as t_{r-ESN}, Equation (5.3) is solved for t_{r-ESN} to obtain:

$$t_{r-ESN} = ln\left(\frac{D_f}{\varphi}\right)\ G_{\beta}^{-1}(r) \qquad \ldots(5.7)$$

Let $t_1, t_2, \ldots, t_n$ denoted as a random sample of size n from the TTF distribution given in Equation (5.6). The likelihood function for the parameters $\mu, \sigma, \lambda, \varphi$ of TTF distribution given by:

$$L(\mu, \sigma, \lambda, \varphi\, ; \boldsymbol{t}) =$$

$$\left(\frac{2}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)^n \prod_{i=1}^{n} \phi\left(\frac{t_i-\mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)\Phi\left(\ \lambda\left(\frac{t_i-\mu\ ln\left(\frac{D_f}{\varphi}\right)}{\sigma\ ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \qquad \ldots(5.8)$$

Equation (5.8) is used to find the posterior distribution of the TTFE with the skew-normal degradation parameter.

5.2.2 Exponential Degradation Model with Log-logistic Degradation Parameter

Based on the cdf of the log-logistic distribution, which is given in Equations (2.12), the cdf of the TTFE, denoted by F_{T-EL}, can be obtained as

$$F_{T-EL}(t) = \frac{1}{1+\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}} \quad ,t > 0 \qquad \text{...(5.9)}$$

By taking the derivative of Equation (5.9) with respect to t, which is

$$\frac{d\left(F_{T-EL}(t)\right)}{dt} = \frac{d\left(\frac{1}{1+\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}}\right)}{dt} \qquad \text{...(5.10)}$$

Then, the pdf of the TTFE with log-logistic degradation parameter, denoted as f_{T-EL}, is given by

$$f_{T-EL}(t) = \frac{-\left(-\frac{\omega}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega-1}\right)}{\left(1+\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}\right)^{2}} \qquad \text{...(5.11)}$$

Then, we have

$$f_{T-EL}(t) = \frac{\left(\frac{\omega}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega-1}}{\left(1+\left(\frac{t}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega}\right)^{2}} \qquad \text{...(5.12)}$$

Equations (5.9) and (5.12) follow the log-logistic distribution where the scale parameter is $\alpha \ln\left(\frac{D_f}{\varphi}\right)$ and the shape parameter is ω, which can be written as:

$$t \sim \log - \text{logistic}\left(\alpha \ln\left(\frac{D_f}{\varphi}\right), \omega\right) \qquad \ldots(5.13)$$

The 100 r^{th} percentile of the TTF distribution, denoted as t_{r-EL}, can be determined by solving Equation (5.9) for t_{r-EL}.

Let,

$$\frac{1}{1+\left(\frac{t_{r-EL}}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{-\omega}} = r \qquad \ldots(5.14)$$

Then,

$$\frac{t_{r-EL}}{\alpha \ln\left(\frac{D_f}{\varphi}\right)} = \left(\frac{1-r}{r}\right)^{\frac{-1}{\omega}} \qquad \ldots(5.15)$$

$$t_{r-EL} = \alpha \ln\left(\frac{D_f}{\varphi}\right)\left(\frac{r}{1-r}\right)^{\frac{1}{\omega}} \qquad \ldots(5.16)$$

Suppose that we have a random sample of size *n* from the TTF distribution with parameters $\alpha \ln\left(\frac{D_f}{\varphi}\right)$ and ω, denoted as $t_1, t_2, \ldots, t_n$. Based on Equation (5.12), we obtain the likelihood function given by:

$$L(\alpha, \omega, \varphi; \boldsymbol{t}) = \left(\frac{\omega}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^n \prod_{i=1}^{n} \frac{\left(\frac{t_i}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega-1}}{\left(1+\left(\frac{t_i}{\alpha \ln\left(\frac{D_f}{\varphi}\right)}\right)^{\omega}\right)^2} \qquad \ldots(5.17)$$

5.3 BAYESIAN MODELING OF TTFE

In this section, the Bayesian approach is presented for skew-normal and log-logistic exponential degradation models. Based on Equation (2.17), the posterior distributions of the TTFE with skew-normal and log-logistic degradation parameters are derived and given in the next two sections.

5.3.1 Posterior Density of TTFE with Skew-Normal Degradation Parameter

Equation (2.17) is applied when informative and non-informative prior distributions are considered for the parameters of TTFE with skew-normal degradation parameter. When considering the two types of informative prior, informative and weakly informative gamma priors are assumed for the shape parameter while all the other parameters of the TTFE are assumed to follow the uniform prior distribution. The non-informative prior involves assuming uniform prior distribution for all parameters of TTFE with skew-normal degradation parameter. The values of the hyperparameters of the prior distributions are assumed the same as given in Chapter 3. Then, it is found that the joint posterior density function of the TTFE with skew-normal degradation parameter, denoted as $\pi_{ESN}(\mu, \sigma, \lambda, \varphi|\boldsymbol{t})$, is proportional to

$$\frac{(\lambda)^{a-1}\, e^{\frac{-\lambda}{b}}}{c\, d\, k\;\, b^{a}\,\Gamma(a)}\left(\frac{2}{\sigma\; ln\left(\frac{D_f}{\varphi}\right)}\right)^{n}\prod_{i=1}^{n}\phi\left(\frac{t_i-\mu\;\, ln\left(\frac{D_f}{\varphi}\right)}{\sigma\;\, ln\left(\frac{D_f}{\varphi}\right)}\right)\Phi\left(\lambda\left(\frac{t_i-\mu\;\, ln\left(\frac{D_f}{\varphi}\right)}{\sigma\;\, ln\left(\frac{D_f}{\varphi}\right)}\right)\right) \qquad \ldots(5.18)$$

The marginal posterior for the parameters μ, σ, λ and φ based on joint posterior in Equation (5.18) is analytically complex, so the MCMC method is implemented to estimate the parameters of the TTFE with skew-normal degradation parameter.

5.3.2 Posterior Density of TTFL with Log-Logistic Degradation Parameter

The Bayesian approach is applied to estimate the parameters and the percentiles of TTFE with log-logistic degradation parameter. The results found are compared with those obtained under the Bayesian approach for skew-normal exponential degradation model. The joint posterior density function of the log-logistic exponential degradation model, denoted as $\pi_{EL}(\omega, \alpha, \varphi|\boldsymbol{t})$, is given as follows:

$$\frac{(\omega)^{a-1}\, e^{\frac{-\omega}{b}}}{c\, k\, b^{a}\, \Gamma(a)} \left(\frac{\omega}{\alpha \ln\left(\frac{D_f}{\varphi}\right)} \right)^{n} \prod_{i=1}^{n} \frac{\left(\frac{t_i}{\alpha \ln\left(\frac{D_f}{\varphi}\right)} \right)^{\omega-1}}{\left(1+ \left(\frac{t_i}{\alpha \ln\left(\frac{D_f}{\varphi}\right)} \right)^{\omega} \right)^{2}} \qquad \ldots(5.19)$$

There is no simple form of expression for the marginal posterior density of the parameters α, ω and φ available for estimating the parameters of the log-logistic exponential degradation model. So, the MCMC is carried out based on the JAGS algorithm to determine the parameter estimates.

5.4 SIMULATION STUDY OF TTFE

The data of the TTFE is simulated for different sample sizes n = 30, 60, and 200 based on Equations (5.6) and (5.12) and the prior distributions considered for all parameters are the same as given in Chapter 4. The MCMC is carried out involving 100000 iterations where 50000 of them are used as burn-in.

5.4.1 Bayesian Analysis for TTFE with Skew-Normal Degradation Parameter Based on Several Different Priors

Based on Equation (5.6), the data of the TTFE under skew normal distribution is simulated with the parameters $\mu \ ln\left(\frac{D_f}{\varphi}\right)$, $\sigma \ ln\left(\frac{D_f}{\varphi}\right)$ and λ for different sample sizes n = 30, 60, and 200. The true values of the parameters μ, σ, λ and φ are assumed equal 1, 2, 3 and 6 respectively. In the simulation, we also consider $D_f = 20$. The distributions of the estimated parameters of the posterior distribution based on Equation (5.18) are generated using the MCMC method under the JAGS algorithm. Based on these distributions, the posterior means for the parameters μ, σ, λ and φ are determined and used for estimating percentiles of the TTF distribution. In section 5.3.1, both informative and weakly informative gamma priors are considered. These two types of priors are indicated based on the choice of the hyperparameters a and b. For the case of the informative prior, it is assumed that a = 2 and b = 2, while for the weakly informative prior a = 0.1 and b = 100. A greater level of uncertainty is reflected by the higher values of the variance, i.e. ab^2 . Additionally, the non-informative prior is presented by assuming that the prior distribution of all the parameters follows uniform distribution. The biasness and standard deviations of the estimated parameters and the percentiles of the TTF distribution are also computed. The results are indicated in Tables 5.1 to 5.3.

Based on Tables 5.1 to 5.3, we find the following:

i) For the small sample size n = 30, in 2 out of 4 cases, the B values of the estimated parameters of TTFE found based on the informative prior are smaller than those found based on the other priors. Also, in 4 out of 5 cases, the B values of the estimated percentiles of TTFE found based on informative prior are smaller than those found based on other prior. For example, for the scale parameter σ, the B values are -0.433, -0.772 and -0.754 under the informative, weakly informative and non-informative priors respectively.

ii) For the small sample sizes n = 30 and 60, in 2 out of 4 cases and in 3 out of 5 cases respectively, the SD values of the estimated parameters and percentiles of TTFE based on the informative prior are found smaller than those found based on the other prior. For example, for sample size n = 60, the SD values for the scale parameter σ are 0.803, 0.857 and 0.829 under the informative, weakly informative and non-informative priors respectively.

iii) For the sample of size n = 60, in 2 out of 4 cases, the B values of the estimated parameters of TTFE found based on the non-informative prior is smaller than those found based on the other priors. In 2 out of 5 cases, the B values of the estimated percentiles of TTFE found for the two priors, which are non-informative and weakly informative priors, are smaller than those found based on the informative prior.

iv) For the large sample size n = 200 and for all priors, the B and SD values of the estimated parameters and the percentiles of TTFE are found close to each other. For example, for the scale parameter σ, the B values are -0.624, -0.527 and -0.568 and the SD values are 0.786, 0.797 and 0.821 under the informative, weakly informative and non-informative priors respectively.

Table 5.1 PE, B and SD for the parameters and certain percentiles of the TTFE with skew-normal degradation parameter under informative, weakly informative and non-informative priors for n = 30

Parameters	Informative prior			Weakly Informative prior			Non-informative prior		
	PE	B	SD	PE	B	SD	PE	B	SD
$\mu = 1$	1.302	-0.302	0.416	0.712	0.288	0.315	0.924	0.076	0.419
$\sigma = 2$	2.433	-0.433	0.818	2.772	-0.772	0.829	2.754	-0.754	0.827
$\lambda = 3$	6.177	-3.177	3.090	41.513	-38.513	53.382	4.501	-1.501	1.108
$\varphi = 6$	7.201	-1.201	2.433	7.386	-1.386	2.494	7.784	-1.784	2.516
$t_{0.05} = 0.811$	1.178	-0.366	0.254	0.768	0.043	0.193	0.634	0.177	0.278
$t_{0.2} = 1.685$	1.837	-0.152	0.204	1.328	0.357	0.163	1.387	0.299	0.220
$t_{0.5} = 2.822$	2.881	-0.059	0.247	2.461	0.361	0.251	2.496	0.326	0.257
$t_{0.75} = 3.974$	4.014	-0.041	0.354	3.729	0.245	0.399	3.687	0.287	0.366
$t_{0.9} = 5.165$	5.188	-0.023	0.495	5.046	0.119	0.570	4.920	0.245	0.509

Table 5.2 PE, B and SD for the estimated parameters and certain percentiles of the TTFE with skew-normal degradation parameter under informative, weakly informative and non-informative priors for n = 60

Parameters	Informative prior			Weakly Informative prior			Non-informative prior		
	PE	B	SD	PE	B	SD	PE	B	SD
$\mu = 1$	1.355	-0.355	0.408	1.452	-0.452	0.392	1.360	-0.360	0.410
$\sigma = 2$	2.466	-0.466	0.803	1.439	0.561	0.857	2.085	-0.085	0.829
$\lambda = 3$	4.322	-1.322	1.554	1.351	1.649	1.467	2.609	0.391	1.198
$\varphi = 6$	6.924	-0.924	2.347	5.263	0.737	2.672	6.984	-0.984	2.481
$t_{0.05} = 0.811$	1.171	-0.360	0.206	0.740	0.072	0.269	0.835	-0.024	0.228
$t_{0.2} = 1.685$	1.952	-0.266	0.175	1.711	-0.026	0.198	1.686	-0.003	0.181
$t_{0.5} = 2.822$	3.078	-0.256	0.191	2.809	0.013	0.192	2.742	-0.080	0.190
$t_{0.75} = 3.974$	4.273	-0.299	0.268	3.771	0.203	0.211	3.759	0.215	0.230
$t_{0.9} = 5.165$	5.510	-0.346	0.375	4.693	0.472	0.291	4.786	0.379	0.322

Table 5.3 PE, B and SD for the estimated parameters and certain percentiles of the TTFE with skew-normal degradation parameter under informative, weakly informative and non-informative priors for *n* = 200

Parameters	Informative prior			Weakly Informative prior			Non-informative prior		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$\mu = 1$	1.359	-0.359	0.394	1.325	-0.325	0.407	1.310	-0.310	0.413
$\sigma = 2$	2.624	-0. 624	0.786	2.527	-0.527	0.797	2.568	-0.568	0.821
$\lambda = 3$	3.422	-0.422	0.741	3.367	-0.367	0.772	3.523	-0.523	0.767
$\varphi = 6$	7.967	-1.967	2.316	7.694	-1.694	2.430	7.727	-1.727	2.495
$t_{0.05} = 0.811$	0.891	-0.080	0.112	0.888	-0.077	0.114	0.897	-0.085	0.110
$t_{0.2} = 1.685$	1.682	0.004	0.093	1.684	0.001	0.094	1.679	0.006	0.093
$t_{0.5} = 2.822$	2.748	0.074	0.103	2.751	0.071	0.103	2.745	0.077	0.103
$t_{0.75} = 3.974$	3.849	0.125	0.135	3.849	0.125	0.134	3.851	0.123	0.135
$t_{0.9} = 5.165$	4.989	0.176	0.191	4.983	0.181	0.191	4.996	0.169	0.193

v) For the informative prior, the SD values of the estimated parameters and percentiles decrease as n increase. Under the non-informative prior, the SD values of the estimated percentiles decrease as n increase and this result is not clear for weakly informative prior.

vi) For all sample sizes, the SD values increase when r^{th} percentiles increase, except for when r = 0.05 and sample size n = 60 in the case of weakly informative prior.

With regard to the choice of the priors, the results found for the small sample size are shown to be dominated by the informative prior while for the large sample size, all the results are shown to be close.

5.4.2 Comparison Between Skew-Normal and Log-logistic Exponential Degradation Models

Based on the Equations (5.18) and (5.19), a comparison between the posterior densities is implemented in terms of B and SD. For the case of skew-normal exponential degradation model, location parameter is assumed equal 1 and not estimated while true values of the other parameters are assumed the same as in Section 5.4.1. The results are provided in Table 5.4.

Based in the Table 5.4, we found that

i) The SD values are decreased when the value of n is increased, clearly appearing under the estimated percentiles of the skew-normal and log-logistic exponential degradation model. For example, under skew-normal exponential degradation model and for the 5th percentile, the SD values are 0.295, 0.253 and 0.116 for sample sizes n = 30, 60 and 200 respectively.

ii) The SD values are increased when the values of r^{th} percentiles are increased except for small value of the r^{th} percentiles under skew-normal exponential degradation model. For example, under skew-normal exponential degradation model and for sample size n = 30, from 5th to 90th percentiles, the SD values are 0.295, 0.245, 0.259, 0.350 and 0.498respectively.

Table 5.4 B and SD of the parameters and the percentiles for the skew-normal and log-logistic exponential degradation models for the simulated data for n = 30, 60 and 200

Parameter values	**Skew-normal exponential degradation model**						**Log-logistic exponential degradation model**					
	***n* = 30**		***n* = 60**		***n* = 200**		***n* = 30**		***n* = 60**		***n* = 200**	
	B	**SD**	**B**	**SD**	**B**	**SD**	**B**	**SD**	**B**	**SD**	**B**	**SD**
Scale	0.506	0.554	-0.672	0.744	-0.421	0.875	-0.398	0.914	-0.456	0.912	-0.403	0.921
Shape	-1.428	2.353	0.388	0.845	-0.260	0.471	0.455	0.402	0.085	0.318	-0.157	0.188
$\boldsymbol{\varphi}$	1.732	1.420	-1.658	1.702	-0.359	2.695	-1.113	2.972	-0.452	2.815	-0.698	2.908
$\boldsymbol{t_{0.05}}$	-0.532	0.295	0.438	0.253	-0.017	0.116	0.179	0.161	-0.051	0.127	-0.067	0.065
$\boldsymbol{t_{0.2}}$	-0.377	0.245	0.312	0.180	0.064	0.062	0.180	0.203	-0.112	0.150	-0.072	0.074
$\boldsymbol{t_{0.5}}$	-0.265	0.259	0.205	0.201	-0.154	0.091	0.080	0.284	-0.225	0.202	-0.059	0.095
$\boldsymbol{t_{0.75}}$	-0.184	0.350	0.151	0.270	-0.261	0.152	-0.160	0.517	-0.387	0.343	-0.026	0.155
$\boldsymbol{t_{0.9}}$	-0.096	0.498	0.116	0.367	-0.374	0.218	-0.691	1.113	-0.660	0.665	0.043	0.286

iii) For different sample sizes, in 18 out of 24 cases, the B values are smaller under the log-logistic exponential degradation model than the skew-normal exponential degradation model in estimated the parameters and the percentiles.

iv) In 15 out of 24 cases, the SD values in estimated the parameters and the percentiles of the TTFE are smaller under skew-normal exponential degradation model than those found under the log-logistic exponential degradation model. For example, under skew-normal exponential degradation model and for sample size n = 30, the SD values of the scale parameter and 75th percentile are 0.554and 0.350 respectively while the SD values for those under log-logistic exponential degradation model are 0.914 and 0.517 respectively.

According to the above notes, the performance of the skew-normal exponential degradation model is more precise than the log-logistic exponential degradation model.

5.5 FATIGUE-CRACK DATA

5.5.1 Description of the Fatigue-Crack Data

The data for four test samples of 2017-T4 aluminium alloy, which is an important material in the plane industry, are obtained by measuring the crack length diffusion under a stress level of 200 MPa. Each sample has ten measurements of the crack length which is recorded at increments of 100000 cycles during the experiment. The critical degradation level is 6 mm. Thus, the failure value is recorded if it is equal or larger than the critical degradation level. The fatigue-crack data is provided in Table 5.5 and Figure 5.1 for more details see Si et al. (2012).

From Figure 5.1, it appears that the data exhibit a curvilinear pattern. In particular, an increase in the rotating cycles contributes to an exponential in the degradation length. Thus, it is reasonable to assume an exponential degradation path to describe the data. The results of the convergence chain of the posterior distribution and some details of data analysis are provided in Tables 5.6 to 5.9.

Table 5.5 The fatigue-crack data for four test samples of 2017-T4 aluminum alloy S1, S2, S3 and S4 recorded at various values of the rotating cycles

Rotating cycle (10^5)	fatigue-crack length (mm)			
	S1	S2	S3	S4
1.5	0.04	0.34	0.32	0.38
1.6	0.32	0.40	0.40	0.50
1.7	0.40	0.58	0.50	0.60
1.8	0.48	0.80	0.70	0.75
1.9	0.50	1.20	1.25	1.10
2.0	1.80	1.80	1.40	1.25
2.1	2.00	2.80	2.00	1.50
2.2	2.60	2.40	3.30	2.00
2.3	5.00	3.40	4.70	3.00
2.4	6.00	7.00	5.60	6.8

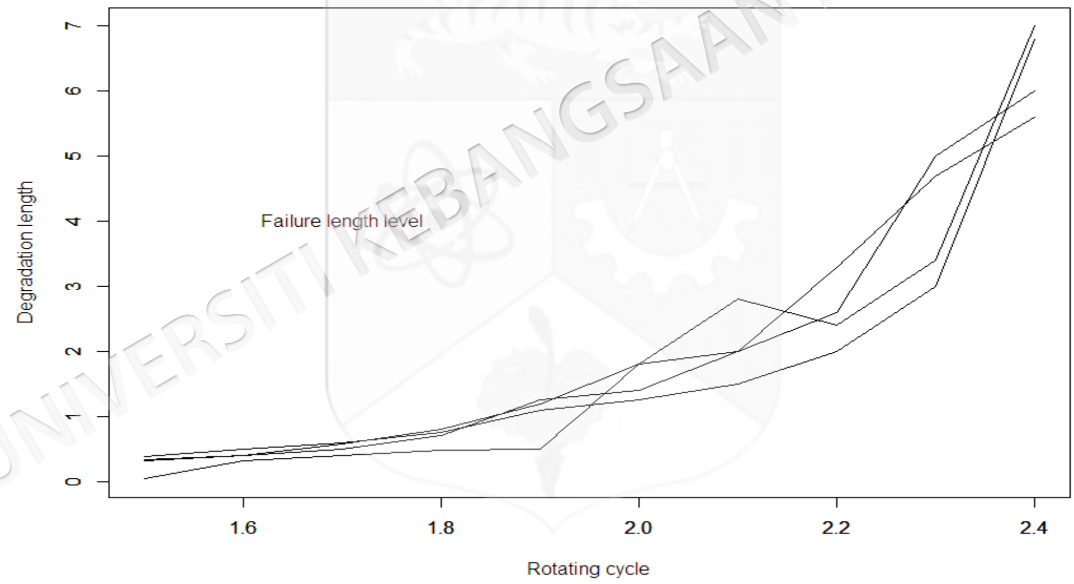

Figure 5.1 The fatigue-crack data of 2017-T4 aluminum alloy

5.5.2 Convergence Analysis Under Fatigue-Crack Data

In order to check if MCMC chain produced using fatigue-crack data based on the JAGS platform converges to the stationary distribution, criteria such as trace plot, autocorrelation factor, Geweke diagnostic and potential scale factor have to be determined and investigated. According to the mechanism of the posterior distribution with the JAGS method of computation, the MCMC chains are assumed to be 2 and for each chain 500000 iterations are run where the first 250000 iterations are treated as burn-in. The values of all the hyperparameters are assumed equal to 3, following the suggestion by Kundu et al. (2020). The graphs of the trace plots, posterior density functions and autocorrelation of the parameters of the TTFE with skew-normal degradation parameter μ, σ, λ and φ are determined and presented in Figure 5.2.

As illustrated in Figure 5.2, based on the trace plots, findings found based on the suggested model converge fairly well. Also, it appears that the auto-correlation graph with thinning of 500 reaches zero quite quickly, indicating that the chain is mixing enough. Furthermore, the convergence case is strengthened by the values of Gelman-Rubin's psrf and Geweke's stationarity test which meet the criteria for convergence, as provided in Table 5.6.

Table 5.6 Geweke's stationarity test and the potential scale reduction factor to assess convergence of the MCMC chains for the parameters μ, σ, λ and φ of the posterior distribution based on fatigue-crack data under skew-normal exponential degradation model

Parameters	**Geweke's stationarity test**			**Psrf**
	Chain 1	**Chain 2**	**Chain 3**	
μ	0.117	-0.334	-0.713	1
σ	0.455	-0.570	-1.441	1
λ	-0.713	0.539	-0.879	1
φ	0.171	-0.376	-0.681	1

A chain is considered to be converging to a stationary distribution if all of the values of the Geweke's stationarity test fall between -1.96 and 1.96 and all of the psrf values are smaller than 1.1. It is clear from Table 5.6 that the outcomes meet the required convergence properties. As a result, the MCMC chains which are generated using the model of interest have successfully converged into a stationary distribution. Finaly, Table 5.7 presents a summary of the posterior distribution for the parameters which consist of mean and standard deviation of the estimated parameters and convergence statistics $\hat{R}$.

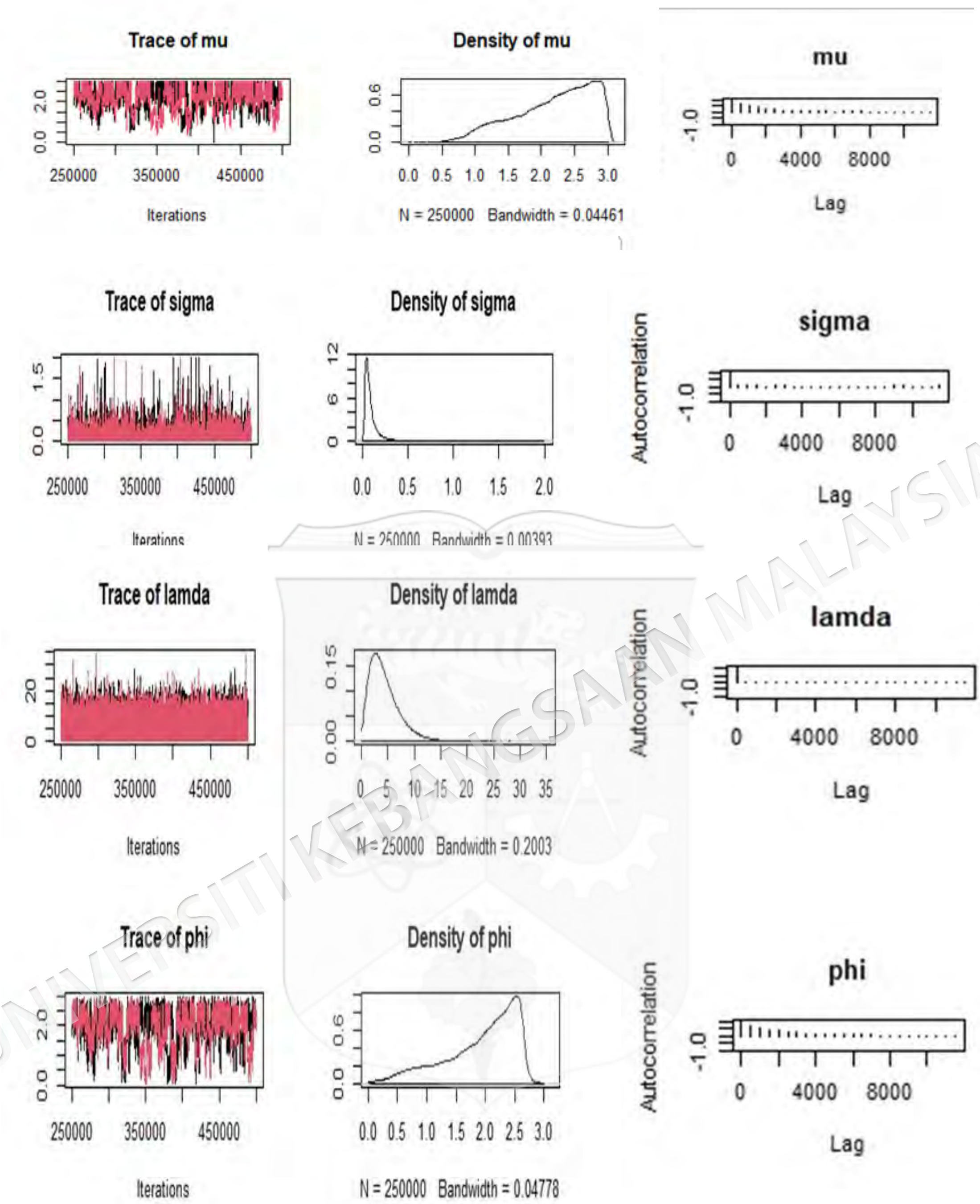

Figure 5.2 Trace plot, posterior density function and autocorrelation of the parameters μ, σ, λ and φ based on the Bayesian analysis of fatigue-crack data under skew-normal exponential degradation model

Table 5.7 Summary of the posterior distribution of TTFE based on the skew-normal degradation parameter with fatigue-crack data

Parameters	**Mean**	**SD**	**Quantiles**					$\hat{\boldsymbol{R}}$
			2.5%	**25%**	**50%**	**75%**	**97.5%**	
μ	2.146	0.588	0.680	1.739	2.237	2.633	2.962	1.001
σ	0.107	0.143	0.020	0.043	0.067	0.113	0.454	1.001
λ	4.473	2.890	0.680	2.389	3.789	5.940	11.720	1.002
φ	1.878	0.632	0.385	1.491	2.028	2.389	2.647	1.001

As stated before, if $\hat{R}$ for all the parameters are found less than 1.1, the resultant posterior distributions attained convergence (Gelman and Hill 2007). According to the Table 5.7, $\hat{R} < 1.1$; thus, the posterior distribution found based on the Bayesian analysis of fatigue-crack data is convergent. In the following subsection, the analysis data under skew-normal degradation model is provided based on the different priors.

5.5.3 Application of Fatigue-Crack Data for the Bayesian Approach of Skew-Normal Exponential Degradation Model based on Different Priors

Fatigue-crack data is applied for the comparison of the performance of different priors in estimating the percentiles of TTFE with skew-normal degradation parameter. Informative and weakly informative priors are considered based on the value of the hyperparameters of gamma prior distribution, where the scale and shape hyperparameters are assumed to be 2 and 2 or 0.1 and 0.01 respectively. While for non-informative prior all the parameters follow uniform distribution U (0, 2). The following Table 5.8 has the results of certain estimated percentiles of TTFE with skew-normal degradation parameter, bias and standard deviation obtained using the fatigue-crack data under the different priors.

Table 5.8 PE, B and SD for the percentiles of skew-normal exponential degradation model based on informative, weakly informative and non-informative priors for fatigue-crack data

Percentiles value	**Informative prior**			**Weakly Informative prior**			**Non-informative prior**		
	PE	**B**	**SD**	**PE**	**B**	**SD**	**PE**	**B**	**SD**
$t_{0.05}$= 2.464	2.406	0.058	0.102	2.384	0.080	0.197	2.357	0.106	0.267
$t_{0.2}$= 2.473	2.444	0.028	0.065	2.433	0.039	0.119	2.420	0.053	0.165
$t_{0.5}$= 2.485	2.498	-0.013	0.055	2.497	-0.012	0.097	2.489	-0.004	0.084
$t_{0.75}$= 2.503	2.553	-0.051	0.099	2.558	-0.055	0.175	2.547	-0.044	0.114
$t_{0.9}$= 2.526	2.609	-0.083	0.159	2.616	-0.090	0.274	2.602	-0.076	0.196

According to the Table 5.8, the results found that

i) In 3 out of 5 cases for the B values, values based on the non-informative prior are found smaller than those found based on the other priors. This is particularly true for large values of the r^{th} percentiles while for the small value of the r^{th} percentiles, the informative prior is better.

ii) All SD values of the estimated percentiles under informative prior are smaller than those based on the other priors.

Generally, the estimated percentiles of the TTFE with skew-normal degradation parameter based on informative prior are found to be more precise than those found based on other priors.

5.5.4 Application of Fatigue-Crack Data in Comparing the Bayesian Approach of Skew-Normal and Log-logistic Exponential Degradation Models

In this subsection, the skew-normal and log-logistic exponential degradation models are applied to the fatigue-crack data and comparison is made on the performance of both models in terms of PE, B and SD under the stated assumption in the previous chapters. In addition, the value of μ is assumed equal to 2, based on rounding of the estimated value of μ in Table 5.7. Also, DIC is determined in order to select the best fitted model to the real datasets. The results of models comparison are provided in Table 5.9.

From Table 5.9, the following results are found:

i) All the B values of the skew-normal exponential degradation model are smaller than those found based on the log-logistic exponential degradation model. For example, for the 5^{th} percentile under skew-normal exponential degradation model, the B value is 0.049 while under the log-logistic exponential degradation model, the B value is 0.955.

Table 5.9 PE and SE for the percentiles of skew-normal and log-logistic degradation model for fatigue-crack data

Percentiles value	**Skew-normal degradation model**			**Log-logistic degradation model**		
	PE	**B**	**SD**	**PE**	**B**	**SD**
$t_{0.05}$= 2.464	2.415	0.049	0.061	1.509	0.955	0.489
$t_{0.2}$= 2.473	2.451	0.022	0.041	1.805	0.668	0.521
$t_{0.5}$= 2.485	2.500	-0.015	0.047	2.132	0.353	0.582
$t_{0.75}$= 2.503	2.551	-0.048	0.081	2.447	0.055	0.693
$t_{0.9}$= 2.526	2.602	-0.076	0.122	2.828	-0.302	0.947
DIC		-19.958			3.628	

ii) All the SD values of the percentiles of the skew-normal exponential degradation model are found to be smaller than those under the log-logistic exponential degradation model. For example, for the 5th percentile under skew-normal exponential degradation model, the SD value is 0.061 while under the log-logistic exponential degradation model, the SD value is 0.489.

iii) The DIC of the skew-normal exponential degradation model is smaller than the DIC of the log-logistic exponential degradation model.

According to the above analysis of fatigue-crack data, the skew-normal exponential degradation model outperformed the log-logistic exponential degradation model in estimating the TTFE.

5.6 CONCLUSION

In this chapter, the Bayesian approach is used to estimate the parameters of the TTF distribution and its percentiles based on the exponential degradation model when the degradation parameter follows the skew-normal distribution or log-logistic distribution. Regarding the prior sensitivity analysis involving informative, weakly informative, and non-informative priors, the informative gamma prior is found to produce more precise results in estimating the parameters and the percentiles of the TTF distribution. In addition, comparison of the performance of the skew-normal and log-logistic exponential degradation models is carried out using the simulated data based on bias and standard deviation, while for the real data, point estimate, standard deviation and deviance information criteria are considered as the criteria for comparison. It is clearly found that the Bayesian approach under the skew-normal exponential degradation model performed better. The skew-normal degradation exponential model is more precise in estimating the parameters of the TTFE and its percentiles.

CHAPTER VI

CONCLUSION AND RECOMMENDATIONS

6.1 INTRODUCTION

There are many studies on the subject of Bayesian analysis for modelling of the time-to-failure distribution based on general degradation models available in the literature. This study has considered a more flexible distribution known as the skew-normal distribution for describing the distribution of the degradation parameter in the general degradation model. Estimation of the parameters and the percentiles of the failure time distribution under linear, power and exponential degradation models are considered based on the Bayesian approach involving informative, weakly informative and non-informative priors. This chapter contains the study summary, discussion on the results found and some ideas for future research.

6.2 STUDY SUMMARY

In this study, we are interested in applying Bayesian method for estimating the parameters and the percentiles of the time-to-failure distribution that is derived based on the linear, power and exponential degradation models. In the modelling, the degradation parameter is assumed to follow the skew-normal distribution. Although, Bayesian methods have often been applied in the analysis of the degradation data, there is not much attention given to the Bayesian methods on the estimation of the parameters of time-to-failure distribution based on general degradation models involving the degradation parameter assumed to follow the skew-normal distribution.

The flexibility brought about by the Bayesian approach as well as the skew-normal distribution have contributed to the choice of the statistical method and the distribution in this study. The flexibility in the approach is due to the availability of prior distributions. While in the case of statistical distribution, skew-normal distribution is flexible in the sense that it can deal with data which are asymmetric and exhibit skewness characteristic.

In addition, this study provides a comparison between the results of Bayesian estimation for skew-normal general degradation models and those found based on Bayesian estimation for log-logistic general degradation models. In the skew-normal general degradation modelling, the degradation parameter is assumed to follow skew-normal distribution while in the log-logistic general degradation modelling the degradation parameter is assumed to follow the log-logistic distribution.

6.3 RESULTS AND DISCUSSIONS

The importance of this study is to conduct the statistical inference for the parameters of the linear, power and exponential skew-normal degradation models by applying the Bayesian method. The Bayesian analysis is implemented for estimating the parameters of these models under several assumptions of the prior distributions. The performance of the Bayesian method is investigated based on posterior means and posterior standard deviation for different sample sizes.

With regards to the Bayesian approach for the linear, power and exponential degradation models with skew-normal distribution assumed as a distribution of degradation parameter, the results found based on the informative, weakly informative gamma prior and non-informative prior are compared under simulated data for different sample sizes and real degradation data. Also, the performance of skew-normal linear, power and exponential degradation models and log-logistic linear, power and exponential degradation models is compared based on simulated data for different sample sizes and real degradation data with posterior means, posterior standard deviation and deviance information criteria.

Regarding the first objectives, based on the simulated data, the results found for comparison of the performance of the Bayesian approach for estimating the parameters and the percentiles of skew-normal linear, power and exponential degradation models under informative, weakly informative and non-informative priors indicate that the informative gamma prior contributes to a more precise parameter estimates than weakly informative gamma prior and non-informative priors, particularly for the case of small sample size. In the case of large sample size, the results are found to be quite close regardless of the choice of priors.

The second contribution provided in the study involves comparison of the Bayesian estimates of the parameters and the percentiles of failure time distributions under skew-normal linear, power and exponential degradation models and log-logistic linear, power and exponential degradation models based on computed point estimate, bias and standard deviation. It can be noted that based on simulated data the Bayesian approach for the skew-normal linear, power and exponential degradation models outperform the Bayesian approach for log-logistic linear, power and exponential degradation models.

The third contribution of this study consists of illustration of the findings above through the application of real data where GaAs laser degradation data for linear degradation model, NASA jet turbofan engine for power degradation model and fatigue-crack degradation data for exponential degradation model. Based on the computed point estimate, bias, standard deviation and deviance information criteria, results of the study are further justified through those real data applications.

6.4 RECOMMENDATIONS FOR FUTURE RESEARCH

In this section, some suggestions of future study are outlined as follows:

i) The method of parameter estimation using the Bayesian approach that has been considered for linear and two non-linear models can be extended to other types of degradation models such as stochastic degradation models.

ii) Instead of applying skew-normal distribution to investigate the performance of Bayesian approach, other families of skew-normal distribution such as closed or extended skew-normal distributions can also be applied.

iii) Other type of non-informative prior such as Jeffrey's prior can be applied in the Bayesian approach.

iv) Other estimation methods such as the maximum likelihood and least squares can also be applied instead of the Bayesian approach for estimating the parameters of the time-to-failure distribution based on the general degradation models.

v) Hierarchical Bayesian method is another choice of method of analysis involving degradation models. This may be possible if one believes, say for example in the case of GaAs laser degradation data, the parameter θ's vary between device to device.

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APPENDIX A

PROBABILITY DENSITY FUNCTION FOR TTFL WITH SKEW-NORMAL DEGRADATION PARAMETER

$$F_{T-LSN}(t) = \Phi\left(\frac{\frac{t}{D_f - \alpha} - \mu}{\sigma}\right) - 2\boldsymbol{T}\left(\frac{\frac{t}{D_f - \alpha} - \mu}{\sigma}, \lambda\right)$$

By taking the derivative of both sides of the above equation with respect to *t,* we have

$$f_{T-LSN}(t) = \frac{1}{(D_f - \alpha)\sigma}\phi\left(\frac{t}{(D_f - \alpha)\sigma} - \frac{\mu}{\sigma}\right) - 2\frac{d}{dt}\boldsymbol{T}\left(\frac{t}{(D_f - \alpha)\sigma} - \frac{\mu}{\sigma}, \lambda\right)$$

By using chain rule to find the derivative of Owen function where

$$h = \frac{t}{(D_f - \alpha)\sigma} - \frac{\mu}{\sigma}$$

and

$$a = \lambda$$

as the following:

$$\frac{d}{d\,t}T(h, a) = \frac{d\,T}{dh} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{(D_f - \alpha)\sigma} \ ,$$

$$\frac{d\,T}{dh} = \frac{1}{2\pi}\int_0^{\lambda} \frac{d}{dh}\frac{e^{-\frac{h^2(1+x^2)}{2}}}{1+x^2}\,dx$$

$$= \frac{1}{2\pi}\int_0^{\lambda} -h\,e^{-\frac{h^2(1+x^2)}{2}}\,dx$$

$$= \frac{-e^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\int_0^{\lambda}\frac{h\,e^{-\frac{h^2x^2}{2}}}{\sqrt{2\pi}}\,dx$$

By substituting $u = h\,x$ which implies to $du = h\,dx$, then

$$\frac{-e^{-\frac{h^2}{2}}}{\sqrt{2\pi}}\int_0^{\lambda}\frac{h\,e^{-\frac{h^2x^2}{2}}}{\sqrt{2\pi}}\,dx = -\phi(h)\int_0^{\lambda h}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,du$$

$$= -\phi(h)\int_0^{\lambda h}\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}}\,du$$

By substituting $z = \frac{u}{\sqrt{2}} \rightarrow dz = \frac{du}{\sqrt{2}}$, then

$$\frac{d\,T}{dh} = -\phi(h)\int_0^{\frac{\lambda h}{\sqrt{2}}}\frac{e^{-z^2}}{\sqrt{\pi}}\,dz$$

$$= -\frac{1}{2}\phi(h)\,\mathrm{erf}\left(\frac{\lambda h}{\sqrt{2}}\right)$$

where $\mathrm{erf}(.)$ is the error function.

$$\frac{d}{d\,t}T(h,a) = -\frac{1}{2(D_f-\alpha)}\phi(h)\,\mathrm{erf}\left(\frac{\lambda h}{\sqrt{2}}\right)$$

Then,

$$f_{T-LSN}(t) = \frac{1}{(D_f - \alpha)\sigma}\phi\left(\frac{t}{(D_f - \alpha)\sigma} - \frac{\mu}{\sigma}\right)\left(1 + \operatorname{erf}\left(\frac{\lambda\big(t - (D_f - \alpha)\mu\big)}{(D_f - \alpha)\sigma\sqrt{2}}\right)\right)$$

$$= \frac{2}{(D_f - \alpha)\sigma}\,\phi\left(\frac{t}{(D_f - \alpha)\sigma} - \frac{\mu}{\sigma}\right)\Phi\left(\lambda\left(\frac{t - (D_f - \alpha)\mu}{(D_f - \alpha)\sigma}\right)\right)$$

APPENDIX B

ESTIMATING THE PARAMETERS AND THE PERCENTILES OF SKEW-NORMAL AND LOG-LOGISTIC LINEAR DEGRADATION MODEL BASED ON SIMULATED DATA

```
## Packages:

library(R2jags)

library(sn)

library(Metrics)

## Specify Generating Values:

mu = 1 ## location skew-normal

sigma = 2 ## scale skew-normal

lamda = 3  ## shape skew-normal

phi = 6 ## fixed-effect parameter

d = 20 ## critical degradation

n.iter = 100000 # number of iteration for mcmc

# value of true quantities:

r = c(0.05, 0.2, 0.5, 0.75, 0.9) ## percentiles

## For true percentiles of TTFL under Skew-normal:

tr = rep(NA, length(r))

for(i in 1:length(r)){

  tr[i] <- (d-phi)*qsn(r[i],mu , sigma, lamda)

}

### simulate data of TTF:

n = 200

tsn = rsn(n,mu*(d-phi), sigma*(d-phi), lamda)

######################### JAGS ###################

############### Weakly informative prior #################
```

```
### Simulate values of parameters:
shape = 0.1
rate = 0.01
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.SNw.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
   *dnorm((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
   *pnorm(lamda*((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
   P[i] <- L[i] / C
   ones[i] ~ dbern(P[i])
   }
   ### Priors
   mu ~ dunif(0,2)
   sigma ~ dunif(0,4)
   lamda ~ dgamma(shape, rate)
   phi~ dunif(0, 12)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","tsn","n","C","d",
            "shape", "rate")
inits_sn <- function (){
```

```
  list (mu = runif(1,0,2),
        sigma = runif(1,0,4),
        lamda = rgamma(1, shape, rate),
        phi = runif(1,0,12))
}
parameters_sn <- c("mu","sigma","lamda", "phi")
out.SNw <- jags(data = data_sn,
                inits = inits_sn,
                parameters = parameters_sn,
                "model.SNw.txt",
                n.chains = 2,
                n.thin = 1,
                n.iter = n.iter,
                n.burnin = (n.iter/2))
out.SNw
### analizes MCMC output:
out_SNw.mcmc = as.mcmc(out.SNw)
out_SNw.per = as.matrix(out_SNw.mcmc)
### Bias, MSE  of parameters:
la.sn.w = sapply(out_SNw.per[,2], mean)
mu.sn.w = sapply(out_SNw.per[,3], mean)
ph.sn.w = sapply(out_SNw.per[,4], mean)
si.sn.w = sapply(out_SNw.per[,5], mean)
## parameters
Bmu = bias(mu, mu.sn.w)
Bsigma = bias(sigma, si.sn.w)
Blamda = bias(lamda, la.sn.w)
```

```
Bphi = bias(phi, ph.sn.w)
Mmu = mse(mu, mu.sn.w)
Msigma= mse(sigma, si.sn.w)
Mlamda = mse(lamda,la.sn.w)
Mphi = mse(phi,ph.sn.w)
cbind(Bmu, Mmu)
cbind(Bsigma, Msigma)
cbind(Blamda, Mlamda)
cbind(Bphi, Mphi)
## PE, B, SD and MSE for the percentiles:
perw= rep(0, length(out_SNw.per[,2]))
pew = rep(0, length(r))
Bw = rep(0,length(r))
sdw = rep(0, length(r))
Mw = rep(0,length(r))
for(i in 1:length(r)){
 for(j in 1:length(per)){
   perw[j]= (d-ph.sn.w[j])*qsn(r[i],mu.sn.w[j],
                    si.sn.w[j], la.sn.w[j])
 }
 pew[i] = mean(perw)
 Bw[i] = bias(tr[i],perw)
 sdw[i] = sd(perw)
 Mw[i] = mse(tr[i],perw)
}
cbind(tr,pew,Bw,sdw,Mw)
###########################################
```

```
############### Informative prior ########################
### Simulate values of parameters:
shape = 2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.SNi.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
   *dnorm((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
   *pnorm(lamda*((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
   P[i] <- L[i] / C
   ones[i] ~ dbern(P[i])
   }
   ### Priors
   mu ~ dunif(0, 2)
   sigma ~ dunif(0,4)
   lamda ~ dgamma(shape, rate)
   phi ~ dunif(0,12)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","tsn","n","C","d",
```

```
        "shape", "rate")
inits_sn <- function (){
  list (mu = runif(1, 0, 2),
    sigma = runif(1,0,4),
      lamda = rgamma(1, shape, rate),
      phi = runif(1,0,12))
}
parameters_sn <- c("mu","sigma","phi","lamda")
out.SNi <- jags(data = data_sn,
        inits = inits_sn,
        parameters = parameters_sn,
        "model.SNi.txt",
        n.chains = 2,
        n.thin = 1,
        n.iter = n.iter,
        n.burnin = (n.iter/2))
out.SNi
### analizes MCMC output:
out_SNi.mcmc = as.mcmc(out.SNi)
out_SNi.per = as.matrix(out_SNi.mcmc)
### Bias & SD of parameters & percentiles Tsn
mu.sn.i = sapply(out_SNi.per[,3], mean)
la.sn.i = sapply(out_SNi.per[,2], mean)
ph.sn.i = sapply(out_SNi.per[,4], mean)
si.sn.i = sapply(out_SNi.per[,5], mean)
## parameters
Bmui = bias(mu, mu.sn.i)
```

```
Bsigmai = bias(sigma, si.sn.i)
Blamdai = bias(lamda, la.sn.i)
Bphii = bias(phi, ph.sn.i)
Mmui = mse(mu, mu.sn.i)
Msigmai = mse(sigma, si.sn.i)
Mlamdai = mse(lamda,la.sn.i)
Mphii = mse(phi, ph.sn.i)
cbind(Bmui, Mmui)
cbind(Bsigmai, Msigmai)
cbind(Blamdai, Mlamdai)
cbind(Bphii, Mphii)
## percentiles
peri= rep(0, length(out_SNi.per[,2]))
pei = rep(0,length(r))
sdi = rep(0,length(r))
Bi = rep(0,length(r))
Mi = rep(0,length(r))
for(i in 1:length(r)){
  for(j in 1:length(peri)){
    peri[j]= (d-ph.sn.i[j])*qsn(r[i],mu.sn.i[j],
                        si.sn.i[j], la.sn.i[j])
  }
  pei[i] = mean(peri)
  sdi[i] = sd(peri)
  Bi[i] = bias(tr[i],peri)
  Mi[i] = mse(tr[i],peri)
}
```

```
cbind(tr,pei,Bi,sdi,Mi)
############################################
############### Non-Informative prior #######################
### Simulate values of parameters:
C <- 10000
Ones = rep(1, n)
## JAGS code:
sink("model.SNn.txt")
cat("model {

  # Likelihood
  for(i in 1:n){
  L[i] <- abs((2/((d-phi)*sigma))
  *dnorm((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
  *pnorm(lamda*((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
  P[i] <- L[i] / C
  ones[i] ~ dbern(P[i])}
  ### Priors
  mu ~ dunif(0,2)
  sigma ~ dunif(0,4)
  lamda ~ dunif(0, 6)
  phi ~ dunif(0,12)
  }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","tsn","n","C","d")
inits_sn <- function (){
```

```
 list (mu = runif(1,0,2),
       sigma = runif(1,0,4),
       lamda = runif(1, 0, 6),
       phi = runif(1,0,12))}
parameters_sn <- c("mu","sigma","phi","lamda")
out.SNn <- jags(data = data_sn,
          inits = inits_sn,
          parameters = parameters_sn,
          "model.SNn.txt",
          n.chains = 2,
          n.thin = 1,
          n.iter = n.iter,
          n.burnin = (n.iter/2))
out.SNn
### analizes MCMC output:
out_SNn.mcmc = as.mcmc(out.SNn)
out_SNn.per = as.matrix(out_SNn.mcmc)
### Bias & MSE of parameters:
la.sn.n = sapply(out_SNn.per[,2], mean)
mu.sn.n = sapply(out_SNn.per[,3], mean)
ph.sn.n = sapply(out_SNn.per[,4], mean)
si.sn.n = sapply(out_SNn.per[,5], mean)
## parameters
Bmun = bias(mu, mu.sn.n)
Bsigman = bias(sigma, si.sn.n)
Blamdan = bias(lamda, la.sn.n)
Bphin = bias(phi, ph.sn.n)
```

```
Mmun = mse(mu, mu.sn.n)
Msigman = mse(sigma, si.sn.n)
Mlamdan = mse(lamda,la.sn.n)
Mphin = mse(phi, ph.sn.n)
cbind(Bmun, Mmun)
cbind(Bsigman, Msigman)
cbind(Blamdan, Mlamdan)
cbind(Bphin, Mphin)
## percentiles
pern= rep(0, length(out_SNn.per[,2]))
pen = rep(0,length(r))
sdn = rep(0,length(r))
Bn = rep(0,length(r))
Mn = rep(0,length(r))
for(i in 1:length(r)){
 for(j in 1:length(pern)){
  pern[j]= (d-ph.sn.n[j])*qsn(r[i],mu.sn.n[j],
                    si.sn.n[j], la.sn.n[j]) }
 pen[i] = mean(pern)
 sdn[i] = sd(pern)
 Bn[i] = bias(tr[i],pern)
 Mn[i] = mse(tr[i],pern)
}
cbind(tr,pen,Bn,sdn,Mn)
########## Comparison SN and LL #########
############### Skew-Normal Linear degradation Model ########
### true values of parameters and initial value:
```

```
mu = 1
sigma = 2
lamda = 3
phi = 6
d = 20
n.iter = 100000
# value of true quantities:
r = c(0.05, 0.2, 0.5, 0.75, 0.9) ## percentiles
## For true percentiles of TTFD under Skew normal :
tr = rep(NA, length(r))
for(i in 1:length(r)){
  tr[i] <- (d-phi)*qsn(r[i],mu , sigma, lamda)}
### simulate time to failure:
n = 200
tsn = rsn(n,mu*(d-phi), sigma*(d-phi), lamda)
######## JAGS data:
shape = 2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.SNi.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
```

```
   *dnorm((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
   *pnorm(lamda*((tsn[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
   P[i] <- L[i] / C
   ones[i] ~ dbern(P[i])}
   ### Priors
   sigma ~ dunif(0,4)
   lamda ~ dgamma(shape, rate)
   phi ~ dunif(0,12)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","tsn","n","C","d",
            "shape", "rate","mu")
inits_sn <- function (){
  list (sigma = runif(1,0,4),
      lamda = rgamma(1, shape, rate),
      phi = runif(1,0,12))
}
parameters_sn <- c("sigma","phi","lamda")
out.SNi <- jags(data = data_sn,
             inits = inits_sn,
             parameters = parameters_sn,
             "model.SNi.txt",
             n.chains = 2,
             n.thin = 1,
             n.iter = n.iter,
             n.burnin = (n.iter/2))
```

```
out.SNi
### analizes MCMC output:
out_SNi.mcmc = as.mcmc(out.SNi)
out_SNi.per = as.matrix(out_SNi.mcmc)
### Bias & SD of parameters & percentiles Tsn
per= rep(0, length(out_SNi.per[,2]))
la.sn.i = sapply(out_SNi.per[,2], mean)
ph.sn.i = sapply(out_SNi.per[,3], mean)
si.sn.i = sapply(out_SNi.per[,4], mean)
## parameters
Bsigmai = bias(sigma, si.sn.i)
Blamdai = bias(lamda, la.sn.i)
Bphii = bias(phi, ph.sn.i)
Msigmai = mse(sigma, si.sn.i)
Mlamdai = mse(lamda,la.sn.i)
Mphii = mse(phi, ph.sn.i)
cbind(Bsigmai, Msigmai)
cbind(Blamdai, Mlamdai)
cbind(Bphii, Mphii)
## percentiles
pei = rep(0,length(r))
sdi = rep(0,length(r))
Bi = rep(0,length(r))
Mi = rep(0,length(r))
for(i in 1:length(r)){
  for(j in 1:length(per)){
    per[j]= (d-ph.sn.i[j])*qsn(r[i],mu,
```

```
                        si.sn.i[j], la.sn.i[j]) }
  pei[i] = mean(per)
  sdi[i] = sd(per)
  Bi[i] = bias(tr[i],per)
  Mi[i] = mse(tr[i],per)}
cbind(tr,pei,Bi,sdi,Mi)
######################## For Log-Logistic ##############
Library (eha)
#### True values:
alfa= 2 ## scale log-logistic
omega = 3 ## shape log-logistic
phi = 6
d = 20
n.iter = 100000
# value of true quantities:
r = c(0.05, 0.2, 0.5, 0.75, 0.9) ## percentiles
tr_ll = rep(NA, length(r))
for(i in 1:length(r)){
  tr_ll[i] <- ((d-phi)* alfa)*((r[i]/(1-r[i]))^(1/omega))}
### simulate data of TTF:
n = 30
tll =  rllogis(n,shape = omega, scale = ((d-phi)*alfa))
######################### JAGS ##################
shape = 2
rate = 0.5
C <- 10000
ones = rep(1, n)
```

```
## JAGS code:
sink("model_LL.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   ones[i] ~ dbern(p[i])
   p[i] <- L[i]/ C
   L[i] <- (omega/(alfa*(d-phi)))
   *((tll[i]/(alfa*(d-phi)))^(omega-1))
   *(((1+(tll[i]/(alfa*(d-phi)))^omega))^(-2))}
   ### Priors
   omega ~ dgamma(shape,rate)
   alfa ~ dunif(0,4)
   phi ~ dunif(0, 12)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_ll = list("ones","tll","n","C","d","shape","rate")
inits_ll <- function (){
  list (alfa= runif(1,0,4),
       omega=rgamma(1,shape,rate),
       phi = runif(1, 0, 12))}
parameters_ll <- c("alfa","omega","phi")
out_LL <- jags(data = data_ll,
              inits = inits_ll,
              parameters = parameters_ll,
              "model_LL.txt",
```

```
        n.chains = 2,
        n.thin = 1,
        n.iter = n.iter,
        n.burnin = (n.iter/2))
out_LL
### analizes MCMC output:
out_LL.mcmc = as.mcmc(out_LL)
out_LL.per = as.matrix(out_LL.mcmc)
### Bias & MSE of parameters
om.ll = sapply(out_LL.per[,3], mean)
al.ll = sapply(out_LL.per[,1], mean)
ph.ll = sapply(out_LL.per[,4], mean)
## parameters
Balfa = bias(alfa, al.ll)
Bomega = bias(omega, om.ll)
Bphi = bias(phi, ph.ll)
Malfa = mse(alfa, al.ll)
Momega = mse(omega, om.ll)
Mphi= mse(phi, ph.ll)
cbind(Balfa, Malfa)
cbind(Bomega, Momega)
cbind(Bphi, Mphi)
## percentiles
perl= rep(0, length(out_LL.per[,1]))
pel = rep(0,length(r))
sdl = rep(0,length(r))
Bl = rep(0,length(r))
```

```
Ml = rep(0,length(r))
## percentiles
for(i in 1:length(r)){
  for(j in 1:length(perl)){
    perl[j]= ((d-ph.ll[j])* al.ll[j])*((r[i]/(1-r[i]))^(1/om.ll[j]))}
  pel[i] = mean(perl)
  sdl[i] = sd(perl)
  Bl[i] = bias(tr_ll[i],perl)
  Ml[i] = mse(tr_ll[i],perl)}
cbind(tr_ll,pel,Bl,sdl,Ml)
```

APPENDIX C

ESTIMATING THE PARAMETERS AND THE PERCENTILES OF SKEW-NORMAL AND LOG-LOGISTIC LINEAR DEGRADATION MODEL BASED ON LASER DEGRADATION DATA

```
#### Important packages:

Library (sn)

Library (eha)

Library (R2jags)

Library (Metrics)

#######Input the data as matrix######

#######Input the degradation values in a vector

degradation_values <- c(250, 500, 750, 1000, 1250, 1500, 1750, 2000,
                        2250, 2500, 2750, 3000, 3250, 3500, 3750, 4000)

###### Combine time points (as vectors) using rbind to form a matrix

t1= c(0.47, 0.71, 0.71, 0.36, 0.27, 0.36, 0.36, 0.46, 0.51,0.41, 0.44,
0.39, 0.3, 0.44, 0.51);

t2= c(0.93, 1.22, 1.17, 0.62, 0.61, 1.39, 0.92, 1.07, 0.93, 1.49, 1, 0.8,
0.74,0.7, 0.83);

t3= c(2.11, 1.9, 1.73, 1.36, 1.11, 1.95, 1.21, 1.42,1.57,2.38, 1.57, 1.35,
1.52, 1.05, 1.29);

t4= c(2.72, 2.3, 1.99, 1.95, 1.77, 2.86, 1.46, 1.77, 1.96, 3, 1.96, 1.74,
1.85, 1.35, 1.52);

t5= c(3.51,2.87, 2.53, 2.3, 2.06, 3.46, 1.93, 2.11, 2.59, 3.84, 2.51, 2.98,
2.39, 1.8, 1.91);

t6= c(4.34, 3.75, 2.97, 2.95, 2.58, 3.81, 2.39, 2.4, 3.29, 4.5, 2.84, 3.59,
2.95, 2.55, 2.27);

t7= c(4.91, 4.42, 3.3, 3.39, 2.99, 4.53, 2.68, 2.78, 3.61,5.25, 3.47, 4.03,
3.51, 2.83, 2.78);

t8= c(5.48, 4.99, 3.94, 3.79, 3.38, 5.35, 2.94, 3.02,4.11,6.26, 4.01, 4.44,
3.92, 3.39, 3.42);

t9= c(5.99, 5.51, 4.16, 4.11, 4.05, 5.92, 3.42, 3.29, 4.6, 7.05, 4.51, 4.79,
5.03, 3.72, 3.78);
```

```
t10= c(6.72, 6.07, 4.45, 4.5, 4.63, 6.71, 4.09, 3.75, 4.91, 7.8, 4.8, 5.22,
5.47, 4.09, 4.11);

t11= c(7.13, 6.64, 4.89, 4.72, 5.24, 7.7, 4.58, 4.16, 5.34, 8.32, 5.2, 5.48,
5.84,4.83, 4.38);

t12= c(8.00, 7.16, 5.27, 4.98, 5.62, 8.61, 4.84, 4.76, 5.84, 8.93, 5.66,
5.96, 6.5, 5.41, 4.63);

t13= c(8.92, 7.78, 5.69, 5.28, 6.04, 9.15, 5.11, 5.16, 6.4, 9.55, 6.2, 6.23,
6.94, 5.76, 5.38);

t14= c(9.49, 8.42, 6.02, 5.61,6.32, 9.95, 5.57, 5.46, 6.84, 10.5, 6.54,
6.99, 7.39, 6.14, 5.84);

t15= c(9.87, 8.91, 6.45, 5.95, 7.1, 10.5, 6.11, 5.81, 7.2, 11.3, 6.96, 7.37,
7.85, 6.51, 6.16);

t16= c(10.9, 9.28, 6.88, 6.14, 7.59, 11, 7.17, 6.24, 7.88, 12.2, 7.42,
7.88, 8.09, 6.88, 6.62);

time_points <- rbind(t1, t2, t3, t4, t5, t6, t7, t8, t9, t10, t11, t12, t13, t14,
t15, t16);

####Combine degradation value time points using cbind

out = cbind(degradation_values, time_points)

############################ Graph data ####################

for (i in 2:14) {

 plot(out[,1], out[,i], type= "l",

     ylim=range(out[,-1]), xlab=NA, ylab=NA)

 par(new=TRUE)

}

plot(out[,1], out[,15], type= "l",

    ylim=range(out[,-1]), xlab="Time in Hours",

    ylab="The Percent Increase ")

#############################################################

## obtain the failure time T:

# scale the time point by 250
```

```
out[,1]= out[,1]/250
d = 5 ## critical degradation
T =rep(NA, (dim(out)[2]-1))
for (i in 1:15) {
  Data = data.frame("y"=out[,1], "x"=out[,(i+1)])
  outnew= lm(y~x,data=Data)
  summary(outnew)
  T[i]=predict(outnew, newdata= data.frame("x"= d))
} ## end of loop
summary(T)
n = length(T)
r = c(0.05, 0.2, 0.5, 0.75, 0.9)
Tr = rep(0, length(r))
Tr = quantile(T, probs = r )
########################### MCMC diagnostic ##########
##################### Posterior SN ###################
### simulate values of parameters:
phi = 0
n.iter = 100000
shape = 0.2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.RSNi.txt")
cat("model {
```

```
    # Likelihood
    for(i in 1:n){
    L[i] <- abs((2/((d-phi)*sigma))
    *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
    *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
    P[i] <- L[i] / C
    ones[i] ~ dbern(P[i])
    }
    ### Priors
    mu ~ dunif(0,2)
    sigma ~ dunif(0,2)
    lamda ~ dgamma(shape, rate)
    }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d",
          "shape", "rate", "phi")
inits_sn <- function (){
  list (mu = runif(1,0,2),
      sigma = runif(1,0,2),
      lamda = rgamma(1, shape, rate))}
parameters_sn <- c("mu","sigma","lamda")
Rout.SNi <- jags(data = data_sn,
            inits = inits_sn,
            parameters = parameters_sn,
            "model.RSNi.txt",
```

```
        n.chains = 2,
        n.thin = 1,
        n.iter = n.iter,
        n.burnin = (n.iter/2))
Rout.SNi
### analysis MCMC output:
Rout_SNi.mcmc = as.mcmc(Rout.SNi)
plot(Rout_SNi.mcmc[, 2:4], trace = TRUE, density = TRUE,
   smooth = FALSE, auto.layout = TRUE)
autocorr.plot(Rout.SNi)
gelman.diag(Rout_SNi.mcmc)
gelman.plot(Rout_SNi.mcmc)
geweke.diag(Rout_SNi.mcmc)
geweke.plot(Rout_SNi.mcmc)
###################### ANALISIS DATA ################
######### Posterior SN based on different priors #######
### informative prior #####
### simulate values of parameters:
phi = 0
n.iter = 100000
shape = 0.2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.RSNi.txt")
```

```
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
   *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
   *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
   P[i] <- L[i] / C
   ones[i] ~ dbern(P[i])}
   ### Priors
   mu ~ dunif(0,2)
   sigma ~ dunif(0,2)
   lamda ~ dgamma(shape, rate)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d",
           "shape", "rate","phi")
inits_sn <- function (){
  list (mu = runif(1,0,2),
     sigma = runif(1,0,2),
     lamda = rgamma(1, shape, rate))}
parameters_sn <- c("mu","sigma","lamda")
Rout.SNi <- jags(data = data_sn,
            inits = inits_sn,
            parameters = parameters_sn,
            "model.RSNi.txt",
            n.chains = 2,
```

```
                n.thin = 1,
                n.iter = n.iter,
                n.burnin = (n.iter/2))
Rout.SNi
### analysis MCMC output:
Rout_SNi.mcmc = as.mcmc(Rout.SNi)
Rout_SNi.per = as.matrix(Rout_SNi.mcmc)
### Bias & SD of percentiles:
per= rep(0, length(Rout_SNi.per[,2]))
la.sn.i = sapply(Rout_SNi.per[,2], mean)
mu.sn.i = sapply(Rout_SNi.per[,3], mean)
si.sn.i = sapply(Rout_SNi.per[,4], mean)
## percentiles
pei = rep(0,length(r))
sdi = rep(0,length(r))
Bi = rep(0,length(r))
Mi = rep(0,length(r))
for(i in 1:length(r)){
  for(j in 1:length(per)){
    per[j]= (d-phi)*qsn(r[i],mu.sn.i[j],
                        si.sn.i[j], la.sn.i[j])}
  pei[i] = mean(per)
  sdi[i] = sd(per)
  Bi[i] = bias(Tr[i],per)
  Mi[i] = mse(Tr[i],per)}
cbind(Tr,pei,Bi,sdi,Mi)
###########################################
```

```
############### Weakly informative prior ###################
### simulate values of parameters:
phi = 0
n.iter = 100000
shape = 0.1
rate = 0.01
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.RSNw.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
   *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
   *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
   P[i] <- L[i] / C
   ones[i] ~ dbern(P[i])}
   ### Priors
   mu ~ dunif(0,2)
   sigma ~ dunif(0,2)
   lamda ~ dgamma(shape, rate)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d",
            "shape", "rate", "phi")
```

```
inits_sn <- function (){
  list (mu = runif(1,0,2),
        sigma = runif(1,0,2),
        lamda = rgamma(1, shape, rate))}
parameters_sn <- c("mu","sigma","lamda")
Rout.SNw <- jags(data = data_sn,
                inits = inits_sn,
                parameters = parameters_sn,
                "model.RSNw.txt",
                n.chains = 2,
                n.thin = 1,
                n.iter = n.iter,
                n.burnin = (n.iter/2))
Rout.SNw
### analizes MCMC output:
Rout_SNw.mcmc = as.mcmc(Rout.SNw)
Rout_SNw.per = as.matrix(Rout_SNw.mcmc)
### Bias, MSE & SD of parameters & percentiles Tsn
per= rep(0, length(Rout_SNw.per[,2]))
la.sn.w = sapply(Rout_SNw.per[,2], mean)
mu.sn.w = sapply(Rout_SNw.per[,3], mean)
si.sn.w = sapply(Rout_SNw.per[,4], mean)
## percentiles
pew = rep(0, length(r))
Bw = rep(0,length(r))
sdw = rep(0, length(r))
Mw = rep(0, length(r))
```

```
for(i in 1:length(r)){
 for(j in 1:length(per)){
  per[j]= (d-phi)*qsn(r[i],mu.sn.w[j],
                    si.sn.w[j], la.sn.w[j])}
 pew[i] = mean(per)
 Bw[i] = bias(Tr[i],per)
 sdw[i] = sd(per)
 Mw[i] = mse(Tr[i],per)}
cbind(Tr,pew,Bw,sdw, Mw)
###########################################
############### Non-Informative prior #######################
### simulate values of parameters:
phi = 0
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.RSNn.txt")
cat("model {
  # Likelihood
  for(i in 1:n){
  L[i] <- abs((2/((d-phi)*sigma))
  *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
  *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
  P[i] <- L[i] / C
  ones[i] ~ dbern(P[i])}
  ### Priors
  mu ~ dunif(0,2)
```

```
   sigma ~ dunif(0,2)
   lamda ~ dunif(0, 2)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d","phi")
inits_sn <- function (){
  list (mu = runif(1,0,2),
        sigma = runif(1,0,2),
        lamda = runif(1, 0, 2))}
parameters_sn <- c("mu","sigma","lamda")
Rout.SNn <- jags(data = data_sn,
                inits = inits_sn,
                parameters = parameters_sn,
                "model.RSNn.txt",
                n.chains = 2,
                n.thin = 1,
                n.iter = n.iter,
                n.burnin = (n.iter/2))
Rout.SNn
### analizes MCMC output:
Rout_SNn.mcmc = as.mcmc(Rout.SNn)
Rout_SNn.per = as.matrix(Rout_SNn.mcmc)
### Bias & SD percentiles Tsn
per= rep(0, length(Rout_SNn.per[,2]))
la.sn.n = sapply(Rout_SNn.per[,2], mean)
mu.sn.n = sapply(Rout_SNn.per[,3], mean)
```

```
si.sn.n = sapply(Rout_SNn.per[,4], mean)
## percentiles
pen = rep(0,length(r))
sdn = rep(0,length(r))
Bn = rep(0,length(r))
Mn = rep(0,length(r))
for(i in 1:length(r)){
  for(j in 1:length(per)){
    per[j]= (d-phi)*qsn(r[i],mu.sn.n[j],
                    si.sn.n[j], la.sn.n[j])}
  pen[i] = mean(per)
  sdn[i] = sd(per)
  Bn[i] = bias(Tr[i],per)
  Mn[i] = mse(Tr[i],per)}
cbind(Tr,pen,Bn,sdn,Mn)
#####################################################
######### Comparison LL & SN linear models #########
###################### Log-logistic model######
######### Informative prior ##############################
### Simulate values of parameters:
n.iter=100000
phi = 0
shape = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
```

```
sink("model.RLLn.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   ones[i] ~ dbern(p[i])
   p[i] <- L[i]/ C
   L[i] <- abs((omega/(alfa*(d-phi)))
   *((T[i]/(alfa*(d-phi)))^(omega-1))
   *((1+(T[i]/(alfa*(d-phi)))^omega)^(-2)))}
   ### Priors
   omega ~ dgamma(shape, rate)
   alfa ~ dunif(0,2)
   }",fill = TRUE)
sink()
## Run JAGS model:
data_ll = list("ones","T","n","C","d", "phi", "shape","rate")
inits_ll <- function (){
  list (alfa= runif(1,0,2),
      omega = rgamma(1, shape, rate))}
parameters_ll <- c("alfa","omega")
Rout.LLn <- jags(data = data_ll,
            inits = inits_ll,
            parameters = parameters_ll,
            "model.RLLn.txt",
            n.chains = 2,
            n.thin = 1,
            n.iter = n.iter,
```

```
        n.burnin = (n.iter/2))
Rout.LLn
### analizes MCMC output:
Rout.LLn.mcmc = as.mcmc(Rout.LLn)
Rout.LLn.per = as.matrix(Rout.LLn.mcmc)
### Bias, MSE & SD of percentiles
per.l= rep(0, length(Rout.LLn.per[,1]))
al.ll.n = sapply(Rout.LLn.per[,1], mean)
om.ll.n = sapply(Rout.LLn.per[,3], mean)
## percentiles
pen = rep(0, length(r))
Bn = rep(0,length(r))
sdn = rep(0, length(r))
Mn = rep(0,length(r))
for(i in 1:length(r)){
  for(j in 1:length(per.l)){
   per.l[j]= (d-phi)*al.ll.n[j]*(r[i]/(1-r[i]))^(1/om.ll.n[j]) }
  pen[i] = mean(per.l)
  Bn[i] = bias(Tr[i],per.l)
  sdn[i] = sd(per.l)
  Mn[i] = mse(Tr[i],per.l)}
cbind(Tr,pen,Bn,sdn,Mn)
############################################################
################### DIC ########################
#### DIC_LL
nLL.l <- function(omega, alfa) {
  outll = sum(dllogis(T,shape = omega,
```

```
                    scale = ((d-phi)*alfa),
                    log = TRUE))
  return(-2* outll)}
Deviance_LL = rep(NA, length(al.ll.n))
for (i in 1:length(al.ll.n)) {
  Deviance_LL[i] = nLL.l(om.ll.n[i], al.ll.n[i])}
DIC_LL = 2*mean(Deviance_LL)- nLL.l(mean(om.ll.n),
                                   mean(al.ll.n))
DIC_LL
#################### Posterior SN ####################
Library (sn)
### Simulate values of parameters:
phi = 0
mu = 2
n.iter = 100000
shape = 0.2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
## JAGS code:
sink("model.RSNi.txt")
cat("model {
   # Likelihood
   for(i in 1:n){
   L[i] <- abs((2/((d-phi)*sigma))
   *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
```

```
    *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
    P[i] <- L[i] / C
    ones[i] ~ dbern(P[i])}
    ### Priors
    sigma ~ dunif(0,2)
    lamda ~ dgamma(shape, rate)
    }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d",
            "shape", "rate", "phi","mu")
inits_sn <- function (){
  list ( sigma = runif(1,0,2),
      lamda = rgamma(1, shape, rate))}
parameters_sn <- c("sigma","lamda")
Rout.SNi <- jags(data = data_sn,
             inits = inits_sn,
             parameters = parameters_sn,
             "model.RSNi.txt",
             n.chains = 2,
             n.thin = 1,
             n.iter = n.iter,
             n.burnin = (n.iter/2))
Rout.SNi
### analysis MCMC output:
Rout_SNi.mcmc = as.mcmc(Rout.SNi)
Rout_SNi.per = as.matrix(Rout_SNi.mcmc)
```

```
### Bias & SD of percentiles:
per.s= rep(0, length(Rout_SNi.per[,2]))
la.sn.i = sapply(Rout_SNi.per[,2], mean)
si.sn.i = sapply(Rout_SNi.per[,3], mean)
## percentiles
pei = rep(0,length(r))
sdi = rep(0,length(r))
Bi = rep(0,length(r))
Mi = rep(0,length(r))
for(i in 1:length(r)){
 for(j in 1:length(per.s)){
  per.s[j]= (d-phi)*qsn(r[i],mu, si.sn.i[j], la.sn.i[j])}
 pei[i] = mean(per.s)
 sdi[i] = sd(per.s)
 Bi[i] = bias(Tr[i],per.s)
 Mi[i] = mse(Tr[i],per.s)}
cbind(Tr,pei,Bi,sdi,Mi)
############################################
############DIC#########
### simulate values of parameters:
phi = 0
n.iter = 100000
shape = 0.2
scale = 2
rate = 0.5
C <- 10000
ones= rep(1, n)
```

```
## JAGS code:
sink("model.RSNi.txt")
cat("model {
  # Likelihood
  for(i in 1:n){
  L[i] <- abs((2/((d-phi)*sigma))
  *dnorm((T[i]- (mu*(d-phi)))/((d-phi)*sigma),0,1)
  *pnorm(lamda*((T[i]- (mu*(d-phi)))/((d-phi)*sigma)),0,1))
  P[i] <- L[i] / C
  ones[i] ~ dbern(P[i])}
  ### Priors
  mu ~ dunif(0,2)
  sigma ~ dunif(0,2)
  lamda ~ dgamma(shape, rate)
  }",fill = TRUE)
sink()
## Run JAGS model:
data_sn = list("ones","T","n","C","d",
          "shape", "rate","phi")
inits_sn <- function (){
 list (mu = runif(1,0,2),
     sigma = runif(1,0,2),
     lamda = rgamma(1, shape, rate))}
parameters_sn <- c("mu","sigma","lamda")
Rout.SNi <- jags(data = data_sn,
          inits = inits_sn,
          parameters = parameters_sn,
```

```
              "model.RSNi.txt",
              n.chains = 2,
              n.thin = 1,
              n.iter = n.iter,
              n.burnin = (n.iter/2))
Rout.SNi
### analysis MCMC output:
Rout_SNi.mcmc = as.mcmc(Rout.SNi)
Rout_SNi.per = as.matrix(Rout_SNi.mcmc)
la.sn.i = sapply(Rout_SNi.per[,2], mean)
mu.sn.i = sapply(Rout_SNi.per[,3], mean)
si.sn.i = sapply(Rout_SNi.per[,4], mean)
nLL.sn = function(mu, sigma, lamda){
  outsn = sum(dsn(T, mu*(d-phi), sigma*(d-phi), lamda,
              log = TRUE))
  return(-2 * outsn)}
Deviance = rep(NA, length(la.sn.i))
for (i in 1:length(la.sn.i)) {
  Deviance[i] = nLL.sn(mu.sn.i[i], si.sn.i[i], la.sn.i[i])}
DIC_sn = 2*mean(Deviance)- nLL.sn(mean(mu.sn.i), mean(si.sn.i),
                         mean(la.sn.i))
DIC_sn
```

APPENDIX D

LIST OF PUBLICATIONS

Ba Dakhn, L.N. Bakar, M.A.A. Tajuddin, R.R.M. & Ibrahim, K. 2024. Bayesian Approach for Estimating the Parameters of the Time-to-Failure Distribution and its Percentiles Under Power Degradation Model. Sains Malaysiana.

Ba Dakhn, L.N. Bakar, M.A.A & Ibrahim, K. 2023. Bayesian Estimation of Time-to-Failure Distribution Based on Skew-Normal Degradation Model: An Application to GaAs Laser Degradation Data. Sains Malaysiana 52(2): 641-653.

Ba Dakhn, L.N. Bakar, M.A.A. Tajuddin, R.R.M. & Ibrahim, K. 2023. Parametric and Semi-parametric Skew-Normal Linear Degradation Model to Estimate Time-to-Failure Distribution and its Percentiles. AIP Conference Proceedings 1678, American Institute of Physics, 5th International Conference on Mathematical Science (ICMS5), UKM.